AN ANALYSIS ON SUB-REGIONAL VOTING SCHEME

by

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Abstract

Voting systems are a very intriguing subject. Different voting schemes have different stabilities in retaining the original results in the presence of noise(the environmental influence that changes voter's decision). A candidate could win an election under some voting schemes, even though he/she has less supporters than another candidate overall. This research introduces how sub-regional and regional voting systems change with the presence of concentrated noise. This paper presents a new model that analyzes the likelihood of sub-regional, regional and national changes in voting decisions under the influences of multiple concentrated-noise blocks. This model breaks down a nation into equal-sized regions, and a region into equal-sized sub-regions based on the idea of deep learning algorithms, which are distributed representations. In the model, sub-region, region and nation correspond to three levels of a hierarchy. An analysis on the voting results of sub-regions to obtain the voting result of the region is completed in this thesis. In the same way, the voting result of a nation can be obtained by analyzing the voting results of regions. This model utilizes the concept of "deep learning" by using knowledge acquired from lower level representations to help define higher level concepts.

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Chapter 1

Introduction

This Chapter presents the motivation and inspiration behind this research. It also outlines the experimental procedure and expected goals as well as the structure of the thesis.

1.1 Preamble

Currently, there are two kinds of election processes which are most popular, namely, direct popular vote and electoral vote. The way that the popular voting system works is by counting the total number of votes in an election, and the candidate that has the majority of votes wins the election. In contrast, the way that the electoral voting system works is through counting the number of winning regions that a candidate receives nationwide. The candidate with the highest number of winning regions wins the election. The candidate with the highest popular vote in each region wins the region. When there is an uneven distribution of population among regions, regions sometimes are assigned points. The candidate who wins the region gets the points assigned to the region. The points for each region are based on the population of the region.

This thesis studies a third election process in which whoever has the majority of votes in a sub-region wins the sub-region, whoever has the majority of winning sub-regions wins the region, and in the end, whoever has majority of winning regions wins the election. This research will help us analyze the stability(the ability of retaining original voting decision under influence of noise) of these three voting structures. In this thesis, I refer to the popular vote as the national voting system, the electoral vote as the regional voting system and the proposed third method as the sub-regional voting system.

1.2 Motivations

Different voting structures have different forms, allowing the voters to express their votes. Most election systems apply the principle of "one person, one vote." Under this principle, every vote is valued with equal weight. However, some voting systems are different; for example, in a corporate election, each voter is assigned votes by the amount of stock owned. There are other causes of inequality in the weighting of votes. For instance, higher rank and authority could weigh more in voting decision. The Electoral College is the system that the United States of America has been using for electing presidents. In this system, there are 538 electors who make the final decision for choosing the president. These 538 electors are assigned to 50 states based on the weight of each state's population. For example, California is the most populated state and has 55 or so electoral votes [12]. North Dakota has a very small population and only has 3 electoral votes [12]. Reasons that they use Electoral College rather than the popular voting system: it creates a buffer between population and the selection of a President; and gives extra power to the smaller states. The Electoral College is in no way a perfect system because sometimes the winner of the Electoral College loses the popular vote by a fair amount of margin, such as in the 2002 American election [13]. The question arises whether this is fair. Many additional elements could affect voting decisions: bribery, threat, peer pressure, and even weather. These external influences will be called the "noise" in this thesis. In the current election process, these noises are big factors that determine the election results, especially when candidates are very close in the poll. Interference of "noise" makes the voting system even more unstable. How can the influence caused by noise be minimized? How can the stability of the election structure be improved? These are the questions addressed in this research [10].

1.3 Background and Literature Review

This section points out the application of voting schemes to artificial intelligence. The discussions on national voting scheme versus regional voting scheme and sub-regional voting scheme versus regional voting scheme is also made. In addition, this section presents the involvement of deep learning method in this research.

1.3.1 Application of Voting Schemes

The reason that different voting schemes are of interest is Direct Popular Vote and Electoral College Vote can be used in various pattern recognitions for the purpose of achieving a better recognition rate. For example, the national voting scheme and the regional voting scheme can be applied to facial recognition by considering a face as a nation. A facial image is partitioned into small regions, and the regional voting scheme is applied in the process of matching. The facial image is recognized based on the number of matching regions. This method significantly improves the facial recognition rate when the facial image is distorted. It is easy to see that there is a great deal of resemblance between the process of facial recognition and the process of election voting schemes. In other words, if it can be proved that a regional voting scheme is more stable than a national voting scheme under noise influences, then it is reasonable enough to conjecture that the conclusions being made for regional and national voting schemes also apply to complicated matching schemes such as facial feature matching. In [4], such a conjecture is made, and is proven through a model.

In this thesis, simulations were performed to demonstrate that a sub-regional voting scheme is even better than a regional voting scheme for a two-candidate vote. If the sub-regional voting scheme shows more stability under a noise environment than regional voting scheme, it is also reasonable to conjecture that the conclusions made for sub-regional and regional voting schemes apply to complicated matching schemes as well.

1.3.2 National Voting vs Regional Voting

In [4], the authors have developed a model that demonstrates the ability of accommodating noise contaminated votes of regional voting schemes and national voting schemes. In the model, it is showed that the performance of noise confinement is improved in a regional model when the number of regions is increased. Performance starts decreasing after the number of regions passes a threshold point, and eventually reaches the same performance as the national voting scheme. This occurs when the number of regions is equal to the number of people in the nation. In other words, there exists a certain regional size that can provide the most stable regional voting scheme and can accommodate the most noise contaminated votes. If the size of a region goes smaller than this point, the stability starts to decrease. Eventually, it will reach the same stability as the national voting scheme when the regional size is one. Therefore, the authors conclude that under the influence of noise, regional matching is always more stable than national matching. A simple flag demonstration is performed below to show the case that regional voting scheme can maintain the nation's original voting decision under a noise environment, although the national voting scheme fails to maintain the original voting result under the same noise enviroment.

1.3.2.1 Flag Demonstration

Example 1 Two sets of white-black flags

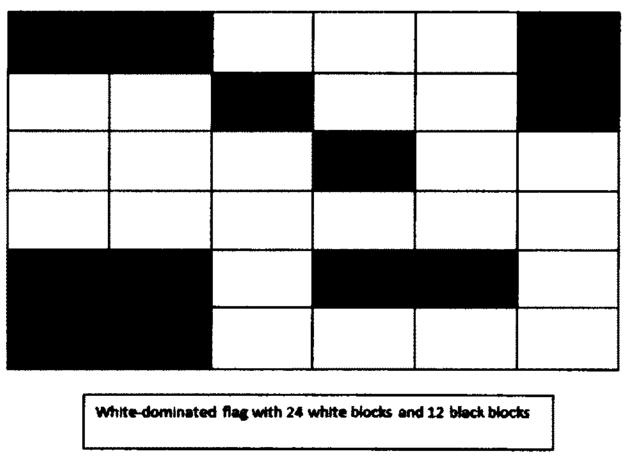


Figure 1.1: Flag Image1

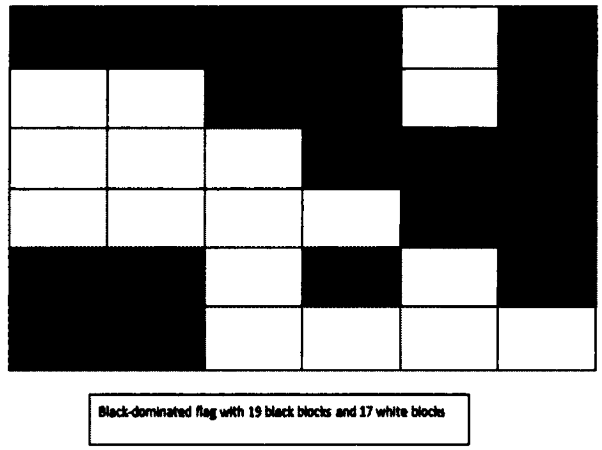


Figure 1.2: Flag Image2

Size of Region	Black Votes	White Votes	Ratio(W/B)
6x6	19	17	0.8947
3x3	1	3	3
2x2	4	5	1.25
1x1	19	17	0.8947

Table 1.1: Table for Flag Experiment

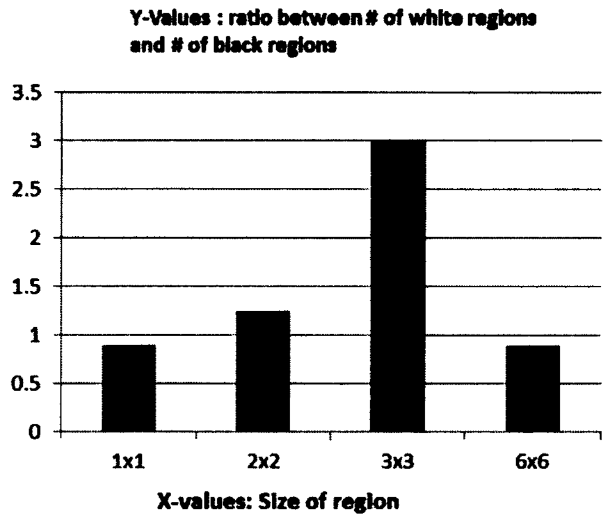


Figure 1.3: Plot for Table 1.1

In the two flag pictures, the flags are either recognized as white-dominated or black-dominated. (In Figure 1.1, each cell denotes a smallest unit of a "pixel"). Here, the flag can be interpreted as a nation. The size of the flag is 6×6 cells. The original white-dominated flag has 24 white cells and 12 black cells(Figure 1.1). When the noise is input into Figure 1.1, Figure 1.1 will be converted into Figure 1.2. When regional voting is applied with a regional size of 3x3 to the flag in Figure 1.2, there are 3 white-dominated regions and 1 black-dominated region. The ratio between the number of white regions and the number of black regions is 3. The flag still remains white-dominated, and the trait of white dominance is even stronger in Figure 1.2 than in Figure 1.1. When only national voting was used on Figure 1.2, then the flag is black-dominated. Therefore, it can be assumed that regional voting would help us sustain the original dominance under the influence of noises. A new question has emerged. How does the size of a region affect the performance of retaining this original dominance? When regional voting is applied with a regional size of 2 × 2 (Figure 1.2), there are 5 white-dominated regions and 4 black-dominated regions. There is still a white-dominated flag here, but the ratio between the number of white regions and the number of black regions is decreased. This demonstrates that there is a certain regional size which provides the best performance of retaining original dominance in the regional voting scheme. If the regional size is selected as 1 × 1, then regional voting would be identical to national voting[3]. Therefore, it is easy to know that the performance of a regional voting scheme is dependent on the size of region. When the regional size decreases from the optimal regional size, this performance decreases until it reaches the same performance as the national voting scheme. The pictures for the regional voting schemes of sizes 3 × 3 and 2 × 2 are displayed in Figure 1.4 and Figure 1.5 respectively.

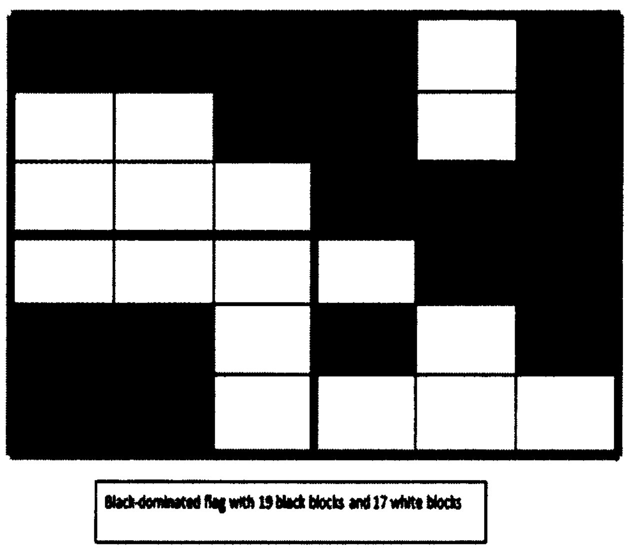


Figure 1.4: Regional Voting Scheme of Size 3x3

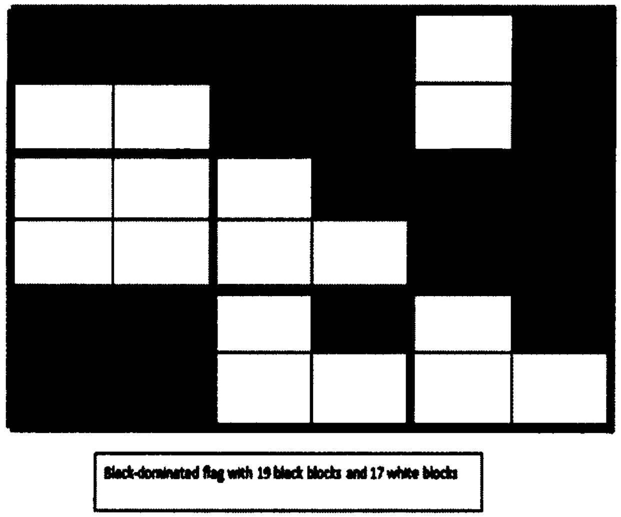


Figure 1.5: Regional Voting Scheme of Size 2x2

1.3.2.2 Terminology

In the thesis, some special terms have been defined, such as "noise", "concentrated-noise", "concentrated-noise contaminated votes","noise-blocks", "convert" and "stability" .

Noise

Noise is defined as environmental influences that would cause voters to change their voting decisions. The name for the noise that causes voters to change their votes from voting for A to voting for B is called pro-B noise. Likewise, the name for the noise that causes voters change their votes from voting for B to voting for A is called pro-A noise. The votes that have been converted under this anti-A noise or anti-B noise are called pro-A contaminated votes or pro-B contaminated votes. Noise can be caused by any type of sources [4].

Concentrated-Noise and Concentrated-Noise Block

Concentrated-noise is defined as a type of noise that affects votes that are within an area of a block. This block is called a concentrated-noise block or noise-block. The noise-block has size $n_l \times n_w$ [4].

Concentrated Noise Contaminated Votes

The votes converted from A to B under concentrated noise are called "anti-A concentrated noise contaminated votes". On the other hand, the votes converted from B to A under concentrated noise are called "anti-B concentrated noise contaminated votes"[4].

Convert

The definition of "convert" in this paper is a change on the voting decision of a sub-region, region, or nation.

Stability

Stability is defined as the ability of retaining original voting decision under the influence of the concentrated noise.

1.3.3 Sub-regional Voting vs Regional Voting

The flag demonstration shows that the regional voting scheme could help maintain original voting results in election processes under the influence of noise. The question arises whether the sub-regional voting scheme would provide a better performance than a regional voting scheme on noise confinement in elections. An example is developed that gives a scenario for which the sub-regional voting scheme is better than the regional voting scheme in performance of noise confinement. This example is similar to the scenario showed in the flag demonstration. A grid as a representation of a nation was used. This grid was composed of white and black blocks. The grid originally was set to be white-dominated, and it had 13 black blocks and 51 white blocks. After the grid was contaminated by the white noise and concentrated noise, the grid appeared to be a black-dominated flag that has 34 black blocks and 30 white blocks. If the black blocks were to stand for the votes of Mitt Romney and the white blocks were to stand for the votes of Barrack Obama, then Mitt Romney would have won the election with the help of noise influence under the national voting scheme. If the grid is broken into four regions, and each region is sized 4×4, then, there would be two regions voting for Mitt Romney and two regions voting for Barrack Obama. Then, the sub-regional voting scheme on this grid can be tried. The grid is broken into four regions of size 4x4, and the regional grid is broken down into four sub-regions of size 2×2. Under this sub-regional voting scheme, there would be two-and-a-half regions voting for Barrack Obama and one-and-a-half regions voting for Mitt Romney. In this case, the sub-regional voting scheme does show better stability in retaining the original election result than the regional voting scheme under the influence of noise. The grid pictures are displayed in Figures 1.6, 1.7, 1.8, and 1.9:

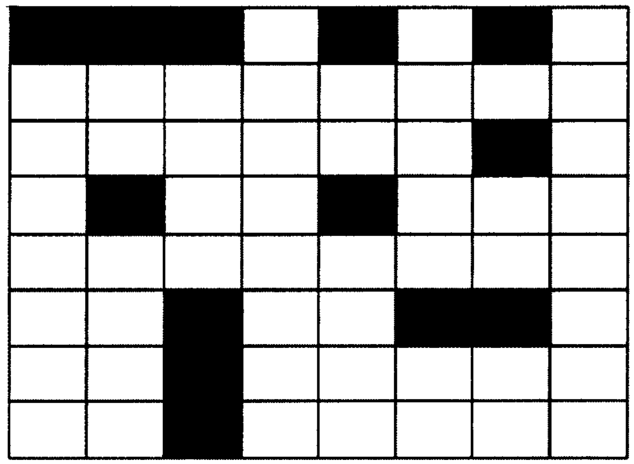


Figure 1.6: The Grid Before The Input of Noise

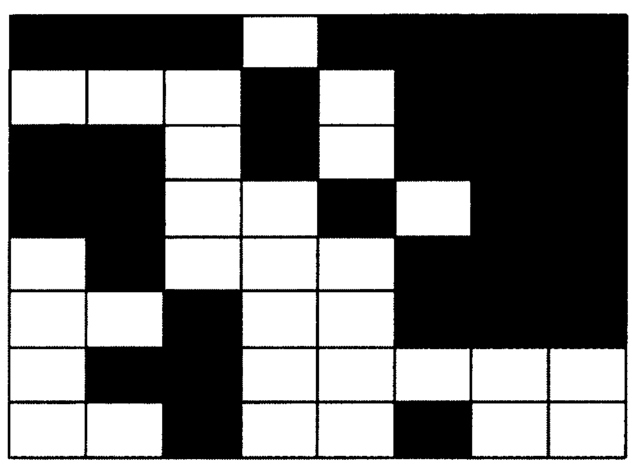


Figure 1.7: The Grid After The Input of Noise

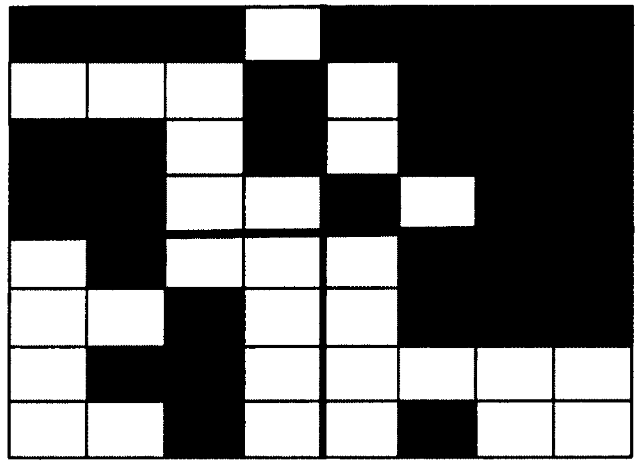


Figure 1.8: Regional Voting Scheme Applied On The Grid

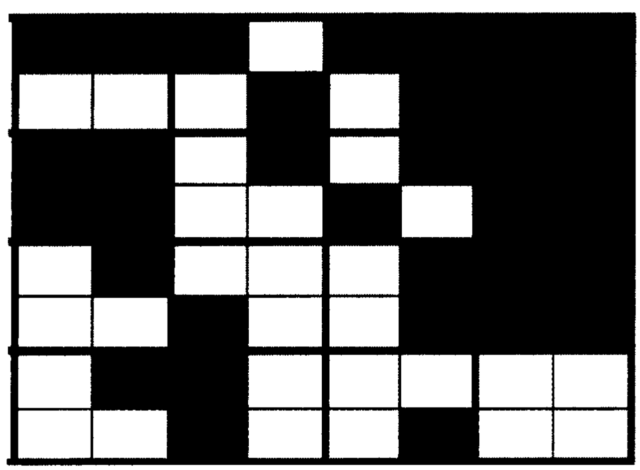


Figure 1.9: Sub-Regional Voting Scheme Applied on The Grid

1.3.4 Deep Learning for AI

What is "Deep Learning" in artificial intelligence? "Deep Learning" is a new area in machine learning . It is a type of learning that uses the lower levels of representations to define the concepts and information of higher level representation in a hierarchical system. The hierarchical system could be a hierarchy of factors, features, or concepts [6]. The reason that "Deep learning" is needed is that an observation can be represented in many different ways. Choosing one representation over another can make a task a lot easier to learn. For example, an image can be represented by using vectors of pixels. Using this representation may help determine whether or not an image is a human's face or not a lot more quickly than other methods. Some of the most successful deep learning methods have been implemented into artificial neural networks. In [5] , it was concluded that "the two-level decoupled Hamming network with middle-sized windows should be a more elegant associative memory model than the one-level Hamming associative memory in all the senses of efficiency, hardware implementation and capacity."[5] The whole process of how the closest memory pattern is selected to the two-level decoupled Hamming memory is based on the deep learning method. In the first level of a two-level decoupled Hamming network, each memory pattern is partitioned into a set of non-overlapping sub-memory patterns. In each sub-memory set, the closest local sub-memory pattern to the sub-memory key is determined by running the Hamming associative memory operations. After all of the sub-memory sets obtain their closest sub-memory patterns to their sub-memory keys, the second level of two-level decoupled Hamming memory (the decision network) applies the voting mechanism on all sets of local sub-memory patterns and generates the closest local memory pattern to the two-level decoupled Hamming memory network [5]. In [1], it is explained that it is important to break down AI problems (such as machine vision or natural language processing) into smaller problems and different levels of distributed representations for the purpose of understanding higher level of abstraction (In computer science, "abstraction" refers to a process of representing data and programs in a form very similar to its meaning, which also reduces the amount of engagements between programmer and tedious implementation details [11]). The author in [1] also introduces the concept of " deep architectures", which are built out of "multiple levels of non-linear operations, such as in neural nets with many hidden layers or in complicated propositional formulae re-using many sub-formulae." One example for "deep architecture" is a mammal's brain. For any given input perceptions, a mammal's brain displays those perceptions into multiple levels of abstraction. Every area of the cortex represents a different level. People like to interpret concepts and ideas in a hierarchical way [1]. The three voting schemes in this thesis have similar resemblances to the "deep architectures" idea. Each voting scheme represents a level of hierarchy. The sub-regional level would be the lowest level in this model, so the voting choice can be determined on a regional level by comparing the numbers of winning sub-regions of the candidates. The process of determining the region's voting preference is a decision making process based on analyzing the sub-regional level's voting choices. Each sub-region is a part of lower level representations for the region, and all sub-regions are distributed with equal weight. In order to determine the national election result, the numbers of winning regions of the candidates have to be compared. This decision making process relies on the voting results from the regional level. Each region would be one of the lower level representations for the nation. Another way to interpret deep learning in this sub-regional model is to treat it as a corporation, because there is a strong analogy between a sub-regional model and a corporate pyramid. In a corporate pyramid, there are generally three levels as well. The general worker is the base level, and the manager is the middle level. The top level of the pyramid is the CEO. We suppose there are 20 workers, 5 managers, and 1 CEO. Every 5 workers reports to one unique manager, and the 5 managers all report to the CEO. The workers gather information and make reports for their managers. The managers make summaries based on the information and reports. The CEO makes a long-term strategic decision based on the suggestions and summarized reports acquired from the managers. Every 5 workers would be similar to one sub-region in the sub-regional model, and every manager corresponds to a region. The CEO's strategic decision is like the presidential election result. The following pictures show the similarities between the corporate model and the sub-regional model.

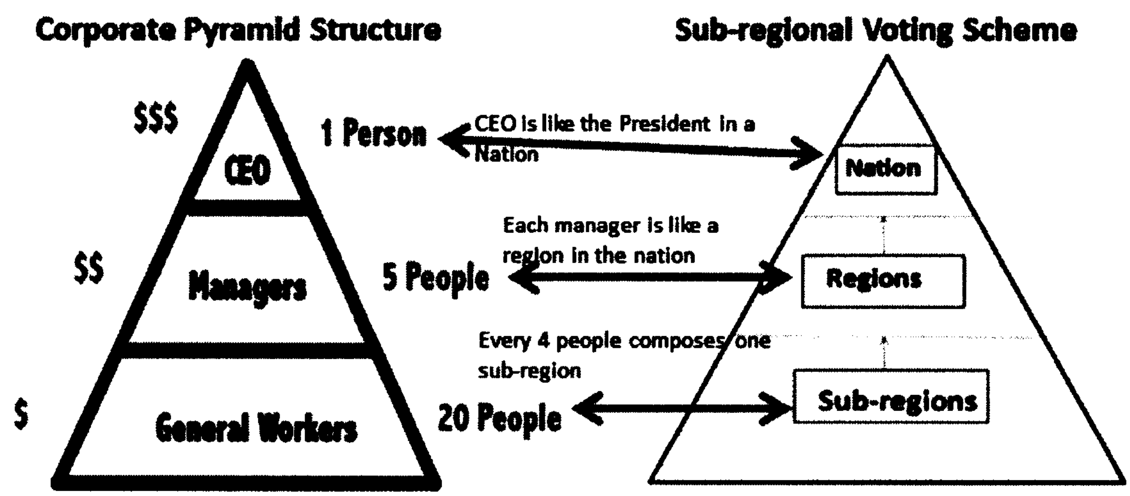


Figure 1.10: Sub-regional Structure vs Corporate Structure

1.4 Overview of The Project

The purpose of this thesis is to present the idea of utilizing the deep learning algorithms-distributed representations on different levels of voting structures.[6] This allows an exploration of the stability difference between regional and sub-regional voting systems with the presence of concentrated noise. Chapter II presents the details of how this model works. Chapter III presents the different experiments conducted in this project. The conclusion of this thesis and discussion of extended work are provided in Chapter IV.

1.5 Contributions

In this thesis, a model that can test the stability of regional voting systems and sub-regional voting systems by using distributed representations is created. This model was tested with different national, regional and sub-regional sizes under the influence of various sizes of noise-blocks. There are several findings:

1. The sub-regional voting system and regional system both become more stable when there is an increase in the size of a nation.

2. Both sub-regional voting systems and regional voting systems become less stable when there is an increase in the size of concentrated noise-blocks.

3. Sub-regional voting systems are more stable than regional voting systems under the same level of concentrated noise.

4. The stability of a sub-regional voting system depends on the size of sub-regions and the size of regions. Some sub-regional size gives the best stability, and some regional size gives the worst stability.

5. Sub-regions in the sub-regional voting scheme will achieve a size with the best stability under concentrated noise for a nation. On the contrary, regions in the sub-regional voting scheme will achieve a size with the worst stability under concentrated noise for a nation.

By studying voting systems, the realization was made that sub-regional voting structure is constructed through the concept of "deep learning". When the process of analyzing higher-level representations is too long to compute or too intricate to look into, the lower level of distributed representations will help understand and interpret the higher level form of representations. It will do this in an easier and better way[6].

Chapter 2

The Model

2.1 Notations

The notations used in the model are listed below:

1. $L \times W$: The nation is composed of $L \times W$ unit cells, where each cell is like an individual person's voting decision. L and W represent the length and width of a nation. They are both positive integers.

2. $n_l \times n_w$: The concentrated noise-block is considered a rectangle with length n_l and width n_w. In this model, $n_l = n_w$, and $n_l \times n_w$ changes from 2×2 to 40×40.

3. $r_l \times r_w$: The size of a region is represented by $r_l \times r_w$, where r_l and r_w represent the length and the width.

4. $s_l \times s_w$: The size of a sub-region is represented by $s_l \times s_w$, where s_l and s_w represent the length and the width.

5. "A" and "B": The assumption is made that there are only two candidates, A and B, in this election. The voting result of a unit cell can only be A or B.

6. "α" and "β": The symbol "α is used to represent the proportion of total votes in the nation supporting A, and the symbol "β" is used to represent the proportion of total votes in the nation supporting B.[1]

7. "α_r" and "β_r": The symbol "α_r" represents the probability of a region voting for A, and the symbol "β_r" represents the probability of a region voting for B.

8. "α_s" and "β_s": The symbol "α_s" represents the probability of a sub-region voting for A, and the symbol "β_r" represents the probability of a sub-region voting for B.

9. $P_N[i, n_w \times n_l]$: the probability of "i" regions being converted from pro-A to pro-B under N concentrated noise-blocks of size $n_w \times n_l$, where "i" changes from 0 to $N \times (\frac{W \times L}{r_w \times r_l})$. "N" is explained in section 2.4.

2.2 Structure of the Model

A nation is represented by a rectangle with an equal size of rectangle-shaped regions and sub-regions. The nation can then be partitioned into regions of size of $r_l \times r_w$, where r_l and r_w are both positive integers and are divisible by L and W respectively. In the sub-regional voting scheme, the sub-partition of each region into equal sizes of rectangle-shaped sub-regions of the size $s_l \times s_w$ is completed. The winner of the national voting scheme is simply whoever gets the majority of votes throughout the nation. The winner of the regional voting scheme is whoever gets

[1]According to "The Law of Large Numbers," as the number of repetitions of the same experiment increases, the proportion of a certain outcome being observed gets closer to the probability of the outcome [8]. A good example is flipping a coin. It is known that the probability of getting heads is 0.5 when the coin is flipped. If more flips are made, in the short run, the proportion of heads would differ from 0.5, but, in the long run, the proportion of getting heads in the outcome gets closer and closer to 0.5. This resembles the model's voting system. When the voting population in a nation is considered reasonably large, α can be treated as the probability of a voter voting for A, and β as the probability of a voter voting for B; hence, $\alpha + \beta = 1$.

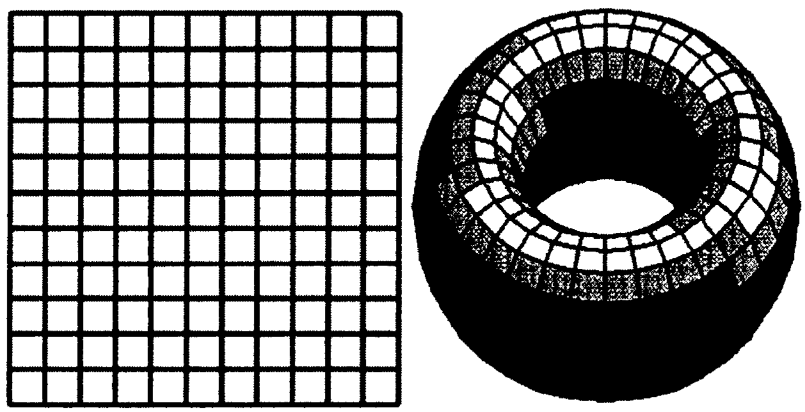
majority of regions in the nation. The sub-regional voting scheme is little different. The process of determining the winner is divided into three steps. First, each sub-region is considered a small nation, and the national voting scheme is used in each sub-region to determine the winner of the sub-region. Then, each region is considered a small nation, but the regional voting scheme is used inside each region to determine the winner of the region. In the end, the winner is determined by the majority of regions in the nation [4].

The value difference between α and β is set very small. It shows the stability difference between the sub-regional scheme and the regional scheme. A small value difference also makes the model as close to a real election situation as possible. Normally, the winner of a presidential election wins by a very small margin. For example, in the 2012 presidential election, Barack Obama won by only 3 %. Therefore, α is set as 50.5%, and β is set as 49.5%. In the following two sections, the model of the sub-regional voting Scheme and the model of the regional voting scheme will be shown. This helps to understand the differences between the sub-regional voting and the regional voting in a nation with the presence of multiple concentrated noise-blocks. In addition, the technique called the extended regional voting method [4] will be mentioned, because it is a crucial concept used in this model.

2.3 Extended Regional Voting

For the purpose of reducing the complication in the computation, the "extended region voting" method is applied as mentioned in [4]. Gluing the edges of a nation together makes the nation look like a ring torus, so that the noise-block is always in the nation as a whole piece. This will create $L \times W$ positions for the concentrated noise-block to be placed in the nation. The lower left corner of the noise-block is always in one region, and the sizes of regions are all the same. Every possible position that a noise-block appears in a nation is exactly the same as one of the positions that a noise block appears in a region. Therefore, the total number of positions that a noise-block can be in is $r_l \times r_w$ instead of $L \times W$. Figure 2.1 shows the shape of a nation and the shape of a nation after gluing the opposing sides.

Figure 2.1: Rectangle and Ring Torus



Nation before gluing Nation after gluing

2.4 Independency of Multiple Noise-Blocks

When there are multiple concentrated noise-blocks appearing in a nation, there could be overlapping areas between concentrated noise-blocks. The overlapping becomes a significant challenge in this model. In order to solve this difficulty, some modifications on voters are used. Each voter is allowed to vote for candidate A or B multiple times. The number of votes that each voter has equals the number of concentrated noise blocks appearing in a nation. For example, each voter is allowed to have 5 votes if there are 5 concentrated noise-blocks in a nation. It is like breaking down the region into 5 layers, with each layer only affected by one unique concentrated noise-block. Another way to look at this relation is to assume that each voter has 5 ballots. Each ballot represents one of the criteria used to select a candidate. The criteria could be any characteristics of candidates such as personality, financial status, policy, physical appearance, sense of humor, etc. Each concentrated noise-block only affects voters' decisions on one criterion. Therefore, there isn't any overlapping problems between concentrated noise-blocks. As such, there exists an independency between concentrated noise-blocks. The size of the nation increases as many times as the number of the concentrated noise-blocks increases, but the sizes of region and sub-region remain the same. If there are only N concentrated noise-blocks, there would be $N \times \frac{L \times W}{r_w \times r_l}$ regions in the nation. A nation is converted by N concentrated noise-blocks if there are more than $N \times (\frac{\alpha_r - \beta_r}{2}) \times (\frac{L \times W}{r_w \times r_l})$ regions being converted in the nation.

2.5 Model of Sub-Regional Voting Scheme on a Nation

To find out how many sub-regions in the region need to be converted in order for the region to be converted from voting for A to voting for B, the calculation of the probability of a sub-region supporting A and the probability of a sub-region supporting B needs to be done. In order to Calculate α_s and β_s, a hypergeometric distribution was used[9]. The hypergeometric distribution is a discrete probability distribution. The probability mass function for the hypergeometric distribution is:

$$P(X = k) = \frac{\binom{K}{k}\binom{M-K}{n-k}}{\binom{M}{n}}. \tag{2.1}$$

This formula calculates the probability of k votes in a sub-region of size of n ($n = s_l \times s_w$) votes for A (or for B) from a region that has a population of M. There are K people voting for A (or for B) inside the region. The sum of k from 0 to K would result in the probability adding up to 1:

$$\sum_{0 \leq k \leq min(n,K)} \frac{\binom{K}{k}\binom{M-K}{n-k}}{\binom{M}{n}} = 1. \tag{2.2}$$

The probability of one sub-region voting for A:

$$\alpha_s = \sum_{\lceil \frac{n}{2} \rceil \leq k \leq n} \frac{\binom{K}{k}\binom{M-K}{n-k}}{\binom{M}{n}}. \tag{2.3}$$

The probability of one sub-region voting for B:

$$\beta_s = 1 - \alpha_s \tag{2.4}$$

Table 2.1 is an example for calculating α_s (the probability of a sub-region voting for A) and β_s (the probability of a sub-region voting for B).

$\alpha = 0.505$	the percentage of votes for A in the region;
$\beta = 0.495$	the percentage of votes for B in the region;
$M = 12 \times 12$	regional size;
$K = round(M \times (\alpha))$	The total number of votes for A in the region; (Number of votes can only be integer, so the value is rounded up(down) since the value could have decimals.)
$n = 4 \times 4$	sub-regional size;
$M - K$	total number of votes for B in the region;
$\alpha_s = \sum_{\lceil \frac{n}{2} \rceil \leq k \leq n} \frac{\binom{K}{k}\binom{M-K}{n-k}}{\binom{M}{n}}$	the probability of sub-region voting for A
$\beta_s = 1 - \alpha_s$	the probability of sub-region voting for B

Table 2.1: Example of calculating α_s and β_s by hypergeometric mass function

Table 2.2 shows the calculated results from table 2.1.

K	71
M	73
α_s	0.6269066267
β_s	0.3730933733

Table 2.2: Calculated results for K,Q,α_s and β_s

Lemma 1. *The minimal number of sub-regions in a region that the concentrated noise needs to convert in order to change the voting decision of this region is computed by*

$$(\frac{\alpha_s - \beta_s}{2}) \times (\frac{r_w \times r_l}{s_w \times s_l}) \tag{2.5}$$

2.5.1 The Pattern of Noise-Block's Appearance in the Sub-Regional Model

In order to compute the probability of sub-region voting decisions, the assumption is made that there is a coordinate system, and the "x" axis of the coordinate system is along the width of the nation, and the "y" axis is along the length of the nation. The origin (0,0) is at the lower left corner of the nation. The way that a noise-block appears within a nation is by the following pattern: assume that the lower left corner of noise block has a coordinate of (x′,y′). The set of all positions of (x′,y′) can be written as:

$$\{(x', y')|x' \in [0, 1, ...W - 1]\, and\; y' \in [0, 1, ...L - 1]\}. \tag{2.6}$$

Let coordinate (C,D) indicate the lower left corner of the region. The set of all positions of (C,D) can be written as:

$$\{(C, D)|C \in [r_w, 2r_w, ...W - r_w]\, and\; D \in [r_l, 2r_l, ..., L - r_l]\}. \tag{2.7}$$

Looking into the position of a noise-block's lower left corner inside of a region, will give the position of the sub-region relative to the lower left corner of the region. If the lower left corner of the region is considered a new origin with coordinate (0,0), then another coordinate (a,b) is needed to represent the lower left corner of the sub-region in which the lower left corner of the noise-block is located. In other words, the lower left corner of the noise-block appears within the sub-region whose lower left corner coordinate is (a,b). The set of all positions of (a,b) can be written as:

$$\{(a,b)|a \in [0, s_w, 2s_w, ...r_w - s_w]\, and\; b \in [0, s_l, 2s_l, ...r_l - s_l]\}. \quad (2.8)$$

Figure 2.2 helps to understand the coordinate systems from a better perspective:

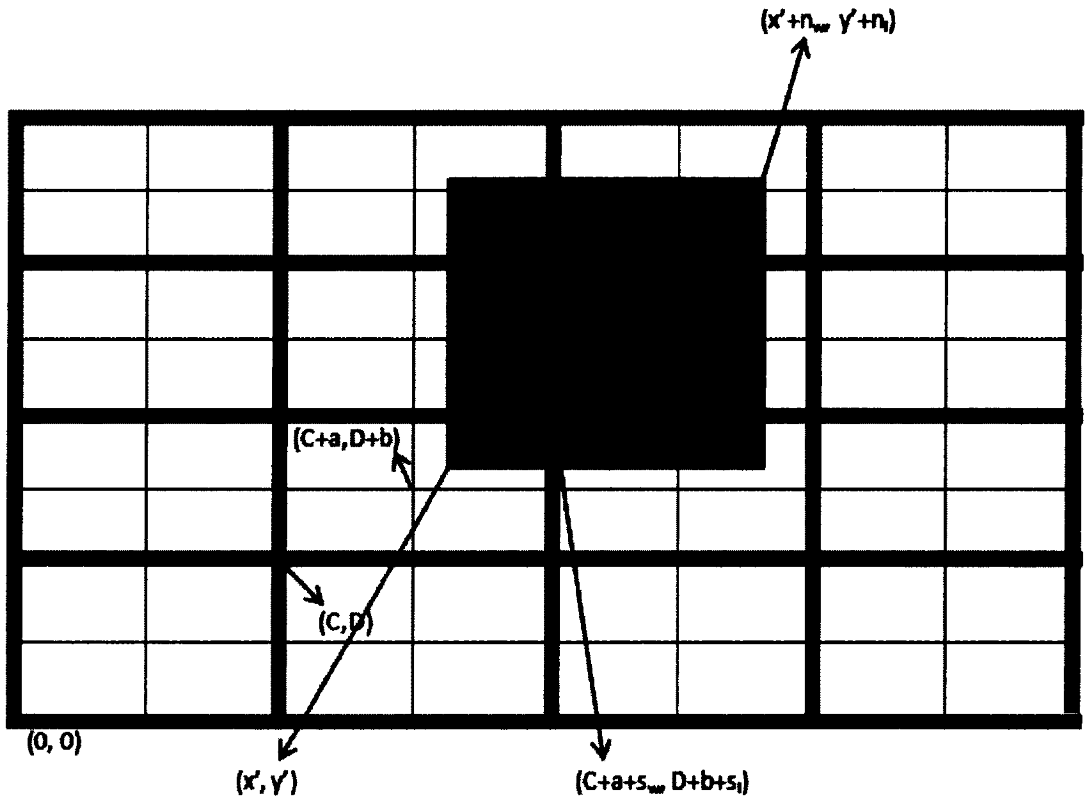


Figure 2.2: One Noise-block appearing in sub-regional model

Theorem 1. *The area in the sub-region being covered by the noise-block is:*

$$\max(0, \min(C+a+s_w, x'+n_w) - \max(x', C+a)) \times \max(0, \min(D+b+s_l, y'+n_l) - \max(y', D+b)), \tag{2.9}$$

where $\max(0, \min(C + a + s_w, x' + n_w) - \max(x', C + a))$ *and* $\max(0, \min(D + b + s_l, y' + n_l) - \max(y', D + b))$ *are the width and length of the overlapping area respectively. The overlapping area is shown as the blue area in Figure 2.4.*

Lemma 2. *The miminal number of votes in a sub-region that the concentrated noise needs to convert in order to change a sub-region from voting for A to voting for B is:*

$$ds = \left(\frac{\alpha - \beta}{2}\right) \times \left(s_w \times s_l\right).$$

This formula gives the condition for the conversion of a sub-region. If the size of the overlapping area is bigger than "ds", then the sub-region is deemed to be compromised by the concentrated noise-block. If the size of overlapping area is less than "ds", then the sub-region retains the original voting decision.

2.5.2 The Probability of a Nation Being Converted by One Concentrated Noise-Block in Sub-Regional Model

Lemma 3. *The miminal number of regions that the concentrated noise needs to convert in order to change a nation from voting for A to voting for B is:*

$$(\frac{\alpha_r - \beta_r}{2}) \times (\frac{L \times W}{r_w \times r_l}). \tag{2.10}$$

If "i" $\in [\lceil(\frac{\alpha_r-\beta_r}{2}) \times (\frac{L\times W}{r_w \times r_l})\rceil, (\frac{W \times L}{r_w \times r_l})]$, *then the nation is converted.*

Theorem 2. *The probability of a nation being converted from pro-A to pro-B is:*

$$\sum_{i=\lceil(\frac{\alpha_r-\beta_r}{2})\times(\frac{L\times W}{r_w \times r_l})\rceil}^{i=\frac{W*L}{r_w * r_l}} P_1[i, n_w \times n_l] = \frac{r[i, n_w \times n_l]}{r_l \times r_w}. \tag{2.11}$$

The ceiling is used because the number of regions has to be an integer.

Theorem 3. *The probability of "i" regions being converted from pro-A to pro-B under one concentrated noise-block is:*

$$P_1[i, n_w \times n_l] = \frac{r[i, n_w \times n_l]}{r_l \times r_w}, \tag{2.12}$$

where "$r[i, n_w \times n_l]$" is the number of times that "i" regions are converted by one noise-block for all noise-block positions in the nation. There are $r_l \times r_w$ positions for one concentrated noise-block to appear in a nation based on the extended region voting method.

2.5.3 The Probability of a Nation Being Converted Under Multiple Concentrated Noise-blocks in Sub-Regional Schemes

The probability of "i" regions being converted from pro-A to pro-B under one concentrated noise-block is:

$$P_1[i, n_w \times n_l] = \frac{r[i, n_w \times n_l]}{r_l \times r_w}. \tag{2.13}$$

The question arises whether the probability $P_k[i, n_w \times n_l]$ would change if $k > 1$. The answer is certainly "yes." If there are two concentrated noise-blocks in a nation, the "i" regions are now converted by the two concentrated noise-blocks. If the first concentrated noise-block converts "$i - q$" regions out of "i" regions, then the second concentrated noise-block converts "q" regions. As mentioned in section 2.4, the number of regions in a nation is increased as many times as the number of the concentrated noise-blocks. For two concentrated noise-blocks, $0 \leq i \leq 2\frac{L \times W}{r_l \times r_w}$. The set of regions that the first concentrated noise-block can convert is

$$\{q | 0 \leq q \leq \frac{W \times L}{r_w \times r_l}\}. \tag{2.14}$$

The set of regions that the second concentrated noise-block can convert is

$$\{i - q | 0 \leq i - q \leq \frac{W \times L}{r_w \times r_l}\}. \tag{2.15}$$

A new set can be derived from 2.15

$$\{q|i - \frac{W \times L}{r_w \times r_l} \leq q \leq i\}. \tag{2.16}$$

Thus, 2.14 and 2.16 can be squeezed into a new set with a smaller interval

$$\{q|Max(0, i - \frac{W \times L}{r_w \times r_l}) \leq q \leq Min(i, \frac{W \times L}{r_w \times r_l})\}. \tag{2.17}$$

Since each concentrated noise-block is independent, the probability of "i" regions being converted from pro-A to pro-B under two concentrated noise-blocks is

$$P_2[i, n_w \times n_l] = \sum_{q=Max(0,i-\frac{W\times L}{r_w\times r_l})}^{Min(i,\frac{W\times L}{r_w\times r_l})} P_1[q, n_w \times n_l] \times P_1[i-q, n_w \times n_l]. \tag{2.18}$$

The probability of "i" regions being converted from pro-A to pro-B by three

concentrated noise-blocks is

$$P_3[i, n_w \times n_l] = \sum_{q=Max(0,i-\frac{W\times L}{r_w\times r_l})}^{Min(i,2\times\frac{W\times L}{r_w\times r_l})} P_2[q, n_w \times n_l] \times P_1[i-q, n_w \times n_l], \tag{2.19}$$

, and we can derive P_N in the same way.

Theorem 4. *The probability of "i" regions converted by N concentrated noise-blocks in the nation is*

$$P_N[i, n_w \times n_l] = \sum_{q=Max(0,i-\frac{W\times L}{r_w\times r_l})}^{Min(i,(N-1)\times\frac{W\times L}{r_w\times r_l})} P_{N-1}[q, n_w \times n_l] \times P_1[i-q, n_w \times n_l]. \tag{2.20}$$

If the nation is converted by N concentrated noise-blocks, then at least $N \times (\frac{\alpha_r-\beta_r}{2}) \times (\frac{L\times W}{r_w\times r_l})$ regions are required to be converted.

Theorem 5. *The probability of a nation being converted from pro-A to pro-B by N concentrated noise blocks is*

$$P = \sum_{i=N\times(\lceil(\frac{\alpha_r-\beta_r}{2})\times(\frac{L\times W}{r_w\times r_l})\rceil)}^{N\times(\frac{W\times L}{r_w\times r_l})} P_N[i, n_w \times n_l]. \tag{2.21}$$

Figure 2.3 is a probability plot drawn from table 2.3. Figure 2.3 shows the stability of the sub-regional voting scheme under the influence of five concentrated noise-blocks. In the figure, the x-axis of this probability plot represents the various sizes of concentrated noise-blocks, and the y-axis represents the probability of a nation being converted. The nation size is set as 120×120, the region size is set as 12×12 and the sub-region size as 4×4. "α" and "β" are set to be 0.505 and 0.495 respectively. In table 2.3, each probability that a nation is converted in the sub-regional scheme corresponds to a certain size of concentrated noise-block.

Size of Noise-Block	Probability of Nation being converted
2	0
3	0
4	0
5	0
6	0
7	0
8	0
9	0
10	0.0282056
11	0.0841537
12	0.18014
13	0.330839
14	0.631613
15	0.815208
16	0.917997
17	0.96875
18	0.983972
19	0.992584
20	0.99702
21	0.999023
22	0.999768
23	0.999969
24	0.999999
25	1
26	1
27	1
28	1
29	1
30	1
31	1
32	1
33	1
34	1
35	1
36	1
37	1
38	1
39	1
40	1

Table 2.3: Data Table for Figure 2.6 and 2.7

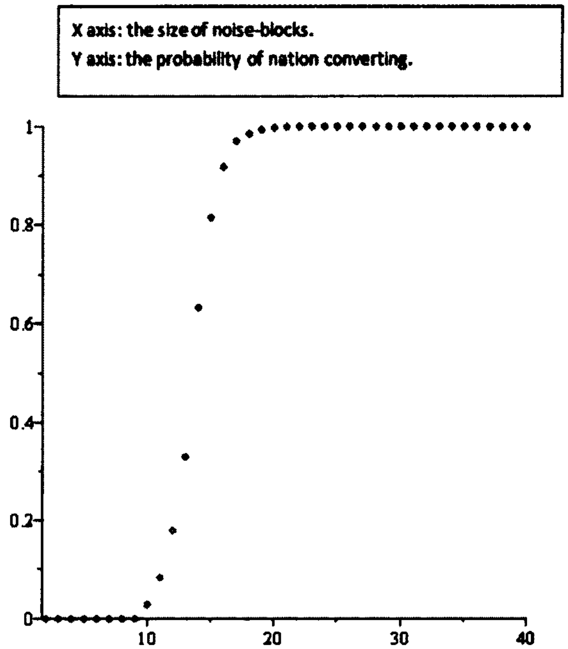


Figure 2.3: 2D Point-Plot of the Stability of Sub-Regional Voting Scheme

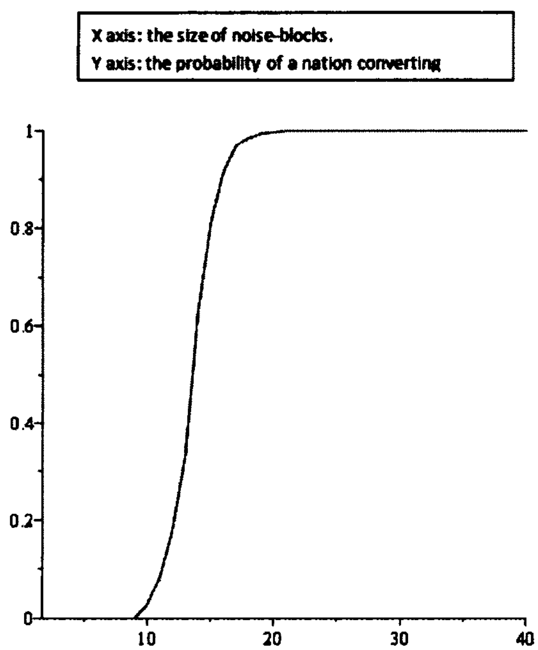


Figure 2.4: 2D Curve-Plot of the Stability of Sub-Regional Voting Scheme

2.6 Model of Regional Voting Scheme on a Nation

The model for the regional voting scheme is very similar to the sub-regional voting scheme, except the part of the computation that involves the sub-regions is omitted. The regional model can be interpreted as another form of the sub-regional model when the sub-regional size is 1×1 or the same as the regional size. In the regional model, the same data for the nation size, region size, α, β, and noise blocks are used for consistency.

2.6.1 The Pattern of the Noise-Block's Appearance in Regional Model

The same assumptions are made that the lower left corner of the noise block has a coordinate of (x′,y′). The set of all positions of (x′,y′) can be written as:

$$\{(x', y')|x' \in [0, 1, ...W - 1] \, and \, y' \in [0, 1, ...L - 1]\}. \tag{2.22}$$

Let coordinates (C,D) indicate the lower left corner of the region in which the lower left corner of the noise block is located. The set of all positions of (C,D) can be written as

$$\{(C, D)|C \in [r_w, 2r_w, ...W - r_w] \, and \, D \in [r_l, 2r_l, ..., L - r_l]\}. \tag{2.23}$$

Theorem 6. *The area in the region being covered by one concentrated noise-block is*

$$\max(0, \min(C+r_w, x'+n_w) - \max(x', C)) \times \max(0, \min(D+r_l, y'+n_l) - \max(y', D)). \tag{2.24}$$

.

The blue area in figure 2.7 indicates the area being covered by one concentrated noise block:

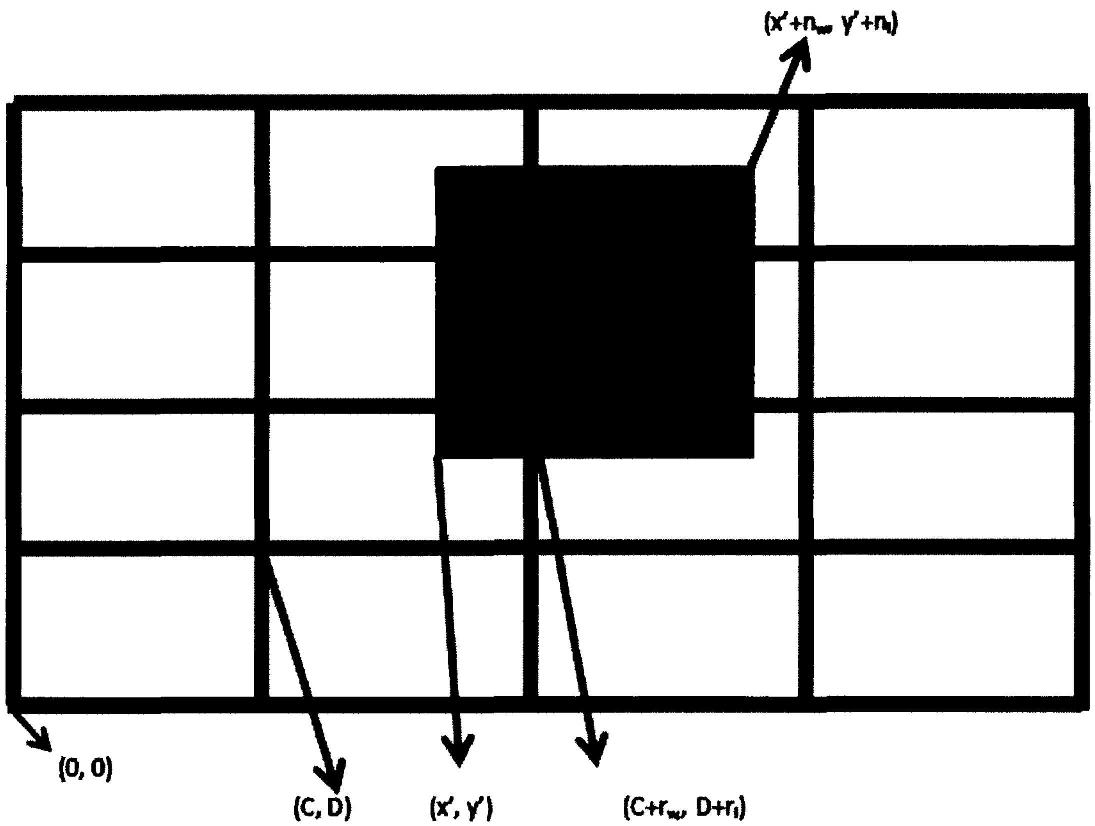


Figure 2.5: One Noise-block appearing in regional model

Lemma 4. *The minimal number of votes that one concentrated noise-block needs to convert in order to change the voting decision of a region is*

$$dr = (\frac{\alpha - \beta}{2}) \times rw \times rl; \tag{2.25}$$

This formula gives the condition for the conversion of a region. If the size of the overlapping area is bigger than "dr", then the region is deemed to be compromised by the concentrated noise-block. If the size of the overlapping area is less than "dr", then the region remains with the original voting decision.

2.6.2 The Probability of a Nation being Converted by One Noise-Block or Multiple Concentrated Noise-blocks in Regional Model

There is no struture difference in regional levels between both models. Therefore, lemma 3 and theorems 3, 4, and 5 are applied again to determine the conversion of a nation by one noise-block and multiple concentrated noise-blocks in the regional model. The calculations are exactly the same as in the sub-regional model's method. The following is the same example used in section 2.5.5.2, except it is in the regional voting scheme. There is no size of sub-region required in the model.

Size of Noise-Block	Probability of Nation being converted
2	2.59454e-05
3	0.000430888
4	0.00226116
5	0.00739857
6	0.0186786
7	0.0400085
8	0.0764859
9	0.134516
10	0.221932
11	0.348111
12	0.524091
13	0.762695
14	0.84641
15	0.904633
16	0.943686
17	0.96875
18	0.983972
19	0.992584
20	0.99702
21	0.999023
22	0.999768
23	0.999969
24	0.999999
25	1
26	1
27	1
28	1
29	1
30	1
31	1
32	1
33	1
34	1
35	1
36	1
37	1
38	1
39	1
40	1

Table 2.4: Data Table for Figure 2.8 and 2.9

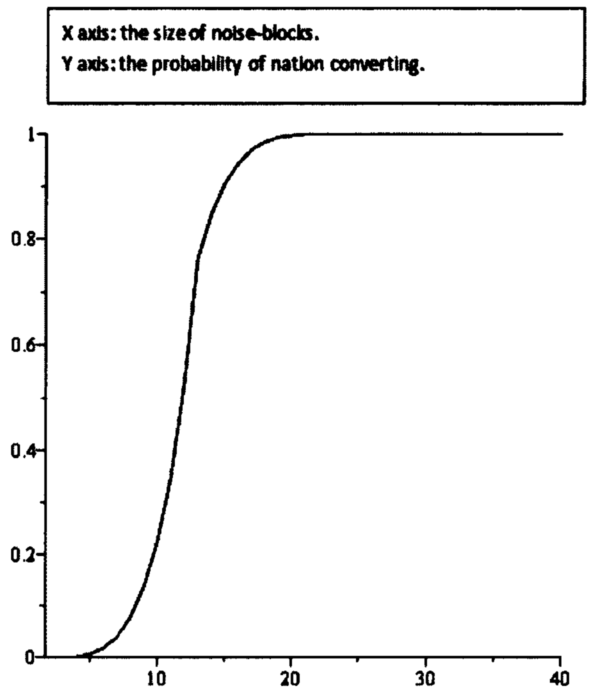


Figure 2.6: 2D Point-Plot of The Stability of Regional Voting Scheme

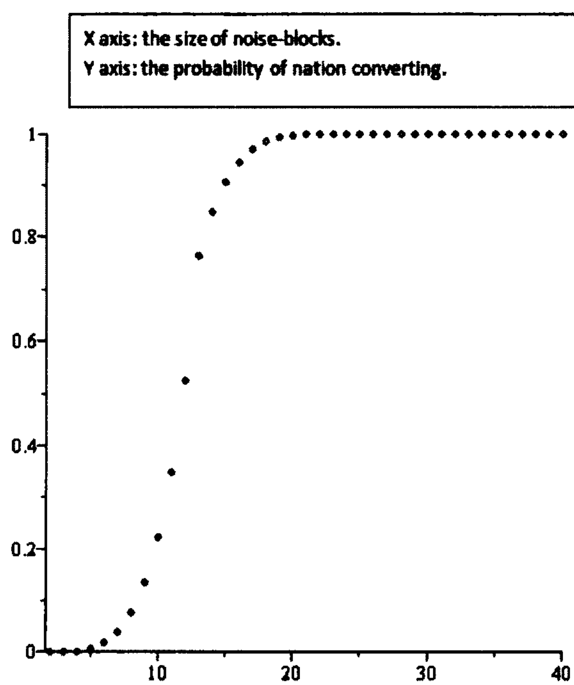


Figure 2.7: 2D Curve-Plot of The Stability of Regional Voting Scheme

2.7 Comparison Between the Sub-Regional Model and the Regional Model

The curves in figure 2.10 represent the probabilities of a nation being converted in the regional voting scheme and the sub-regional voting scheme respectively. Based on the observations in the plots, the same noise influence is more likely to convert the national election results under the regional voting scheme than the sub-regional voting scheme. Both schemes show that the stability of sub-regional voting changes as the size of concentrated noise-blocks change. When the sizes of concentrated noise-blocks get bigger than 10, the probability of the nation being converted is rapidly increased, and the probability curve starts to converge toward the limit y=1.

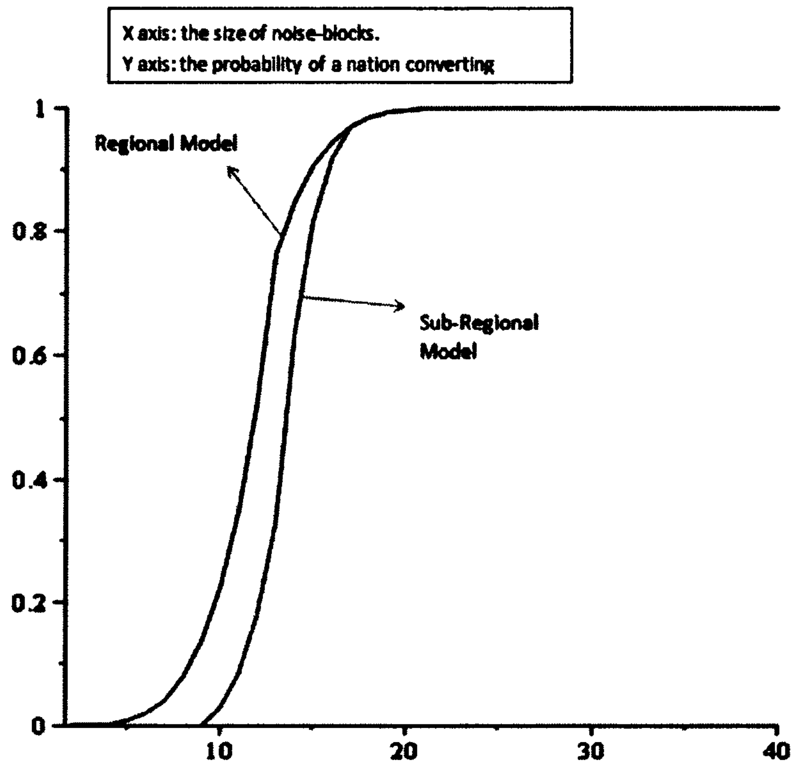


Figure 2.8: The Combination Of 2D Curve-Plots for Sub-Regional and Regional Voting Scheme

Chapter 3

Experiments

3.1 Regional Voting Model

3.1.1 Fixed National Size with Different Regional Sizes and Noise-Block Sizes

In the regional voting model, 120×120 and 160×160 were selected as the sizes of the nation for testing. In each nation, there were only two candidates (A and B). The proportion of votes for A was set as 50.5% and the proportion of votes for B was set as 49.5%. Inside of each nation, there were five equally sized concentrated noise-blocks. The concentrated noise only served in favor of candidate B, since candidate A had a high proportion of supporters. The sizes of the five concentrated noise-blocks changed from 2x2 to 40x40. The stability of the regional voting model was tested under 39 different sizes of five concentrated noise-blocks. The even common divisors (2×2, 4×4, 8×8, 10×10, 20×20 and 40×40) of two testing national sizes were selected as the targeted regional sizes. The comparison could be easily made between two experiments. The reason for choosing even regional sizes was to avoid the possibility that half of the votes occured in the process of counting votes. There were 39 different points at each targeted regional size, and each point represented the probability of the nation being converted at a different size of the concentrated noise-blocks. In figure 3.1 and 3.3, the performance of noise confinement improved as the regional size decreased. In addition, the performance of noise confinement in the regional voting model improved as the size of the noise-blocks decreased.

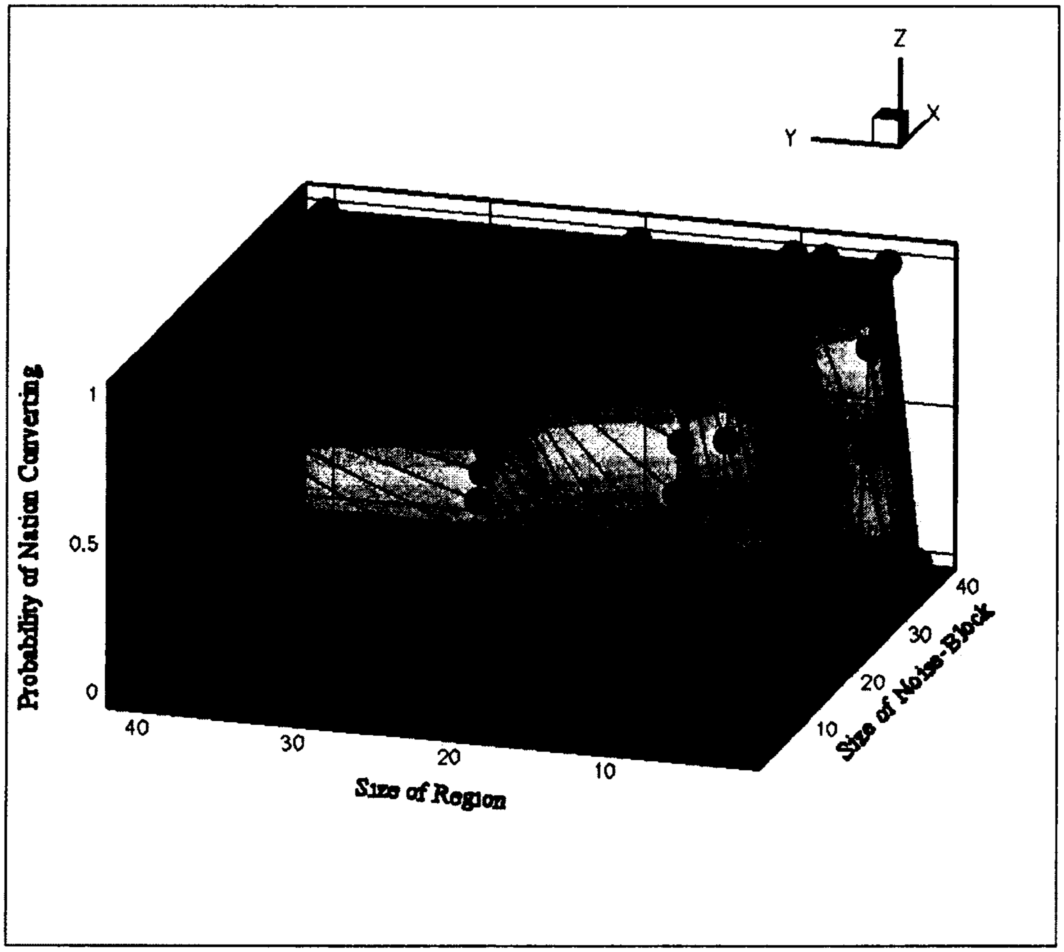


Figure 3.1: National Size of 120 × 120 with Different Regional Sizes

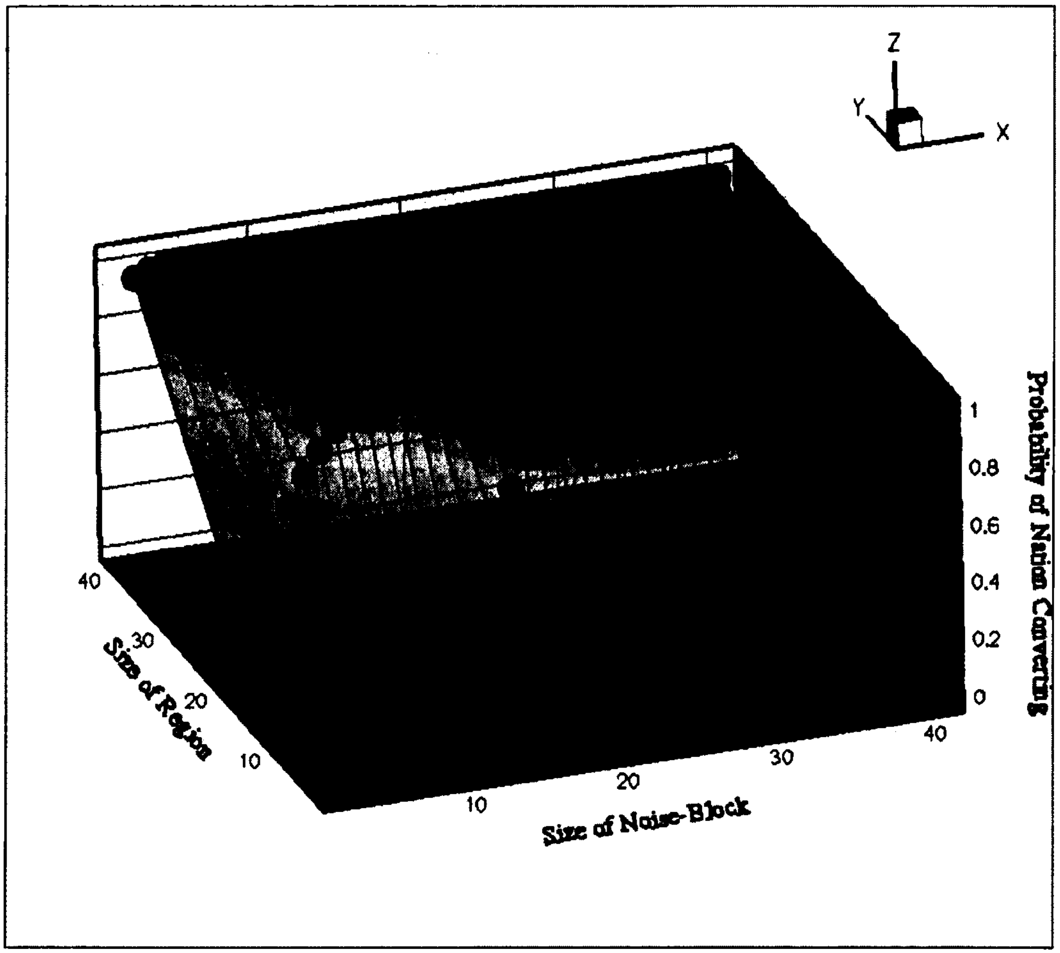


Figure 3.2: National Size of 120 × 120 with Different Regional Sizes at a Different Angle

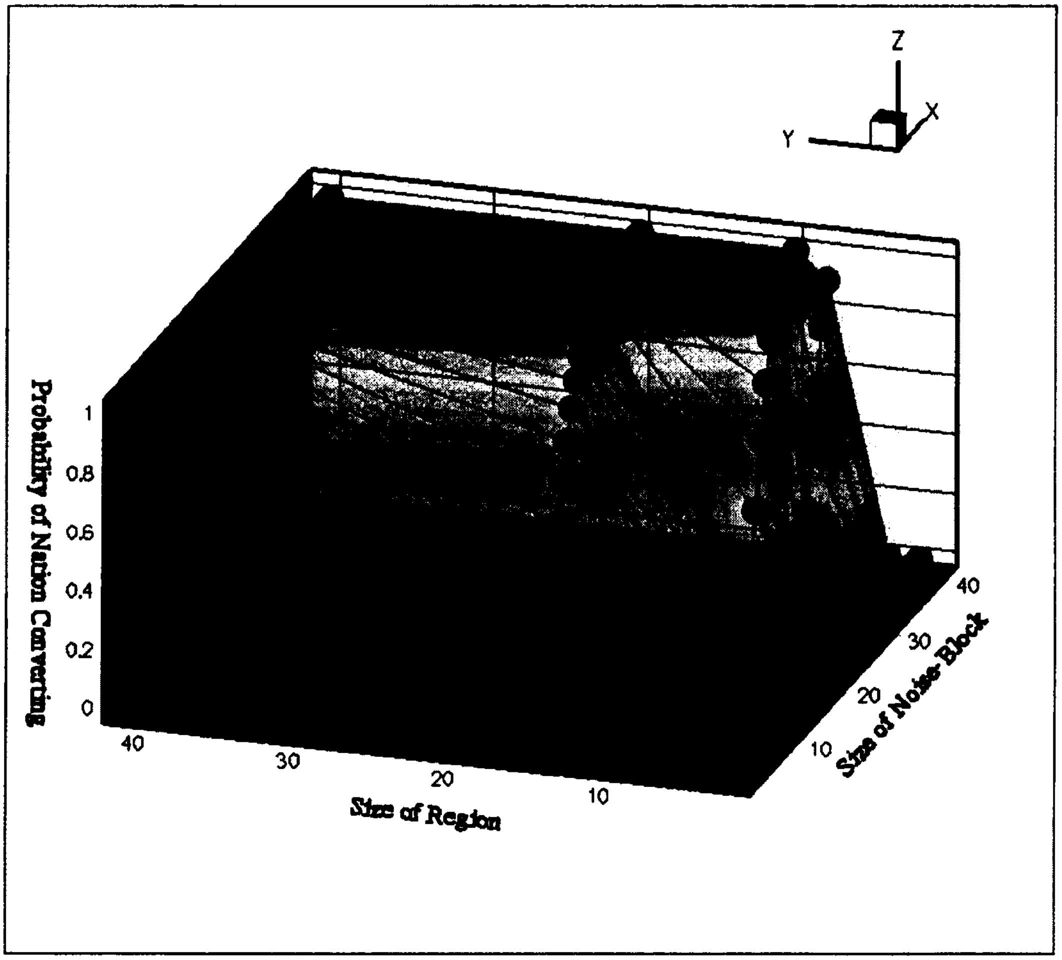


Figure 3.3: National Size of 160 $\times$ 160 with Different Regional Sizes

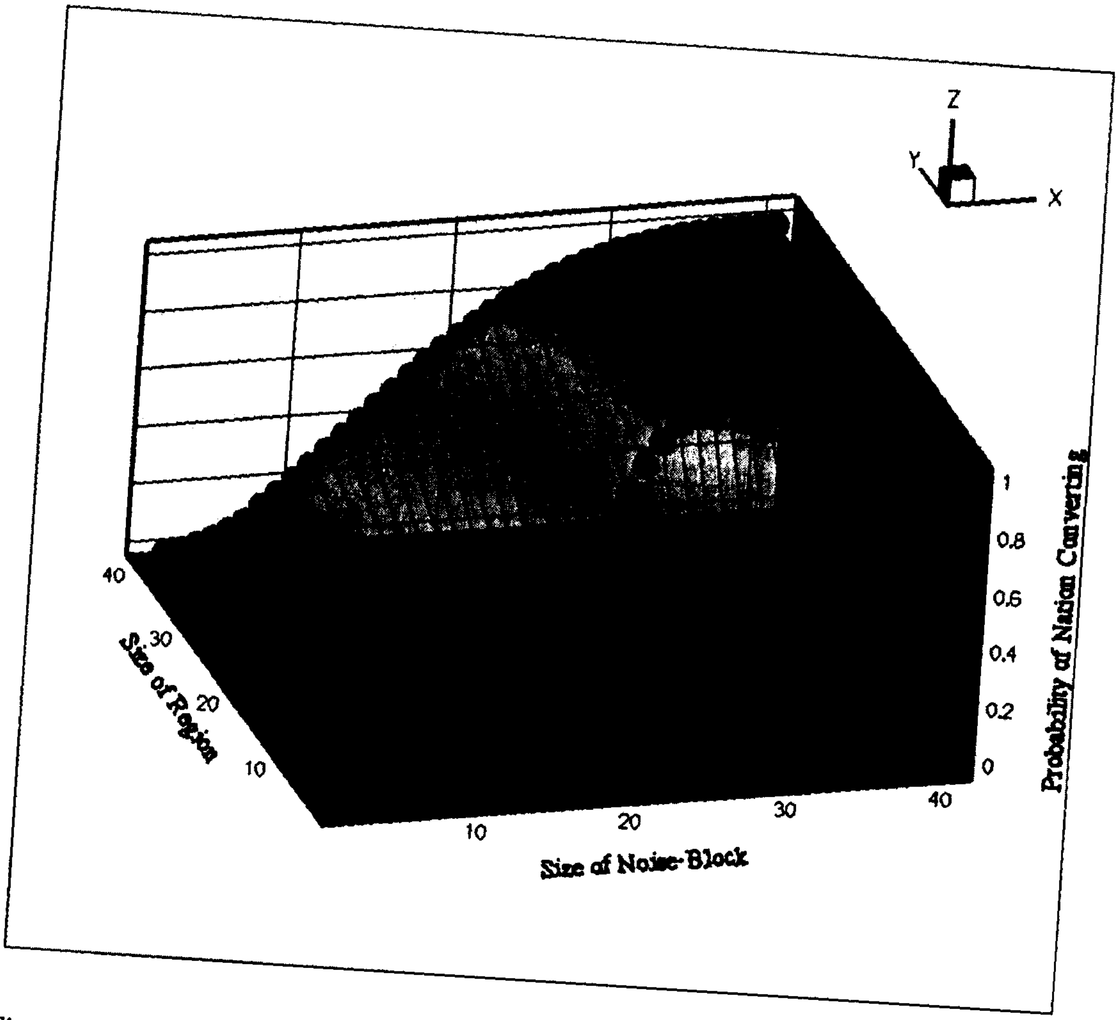


Figure 3.4: National Size of 160 × 160 with Different Regional Sizes at a Different Angle

3.1.2 Comparison between Models That Have Different National Sizes

According to the figures in section 3.1.1, the different national sizes demonstrate the different performances of noise confinement in the regional voting model. Figures 3.5 and 3.6 demonstrate how the performance of noise confinement in the regional model varied with different national sizes. In these figures, the additional test for a national size of 200×200 was added. It indicated that there was 0 probability of the nation being converted at all 39 different sizes of concentrated noise-blocks for all targeted regional sizes in the regional model when the national size was set as 200×200. The green colour stands for the national size of 120×120 in the regional model, the red colour stands for the national size of 160×160, and the blue colour stands for the national size of 200×200. In these two figures indicate that the peformance of noise confinement improved as the national size decreased in the regional voting model.

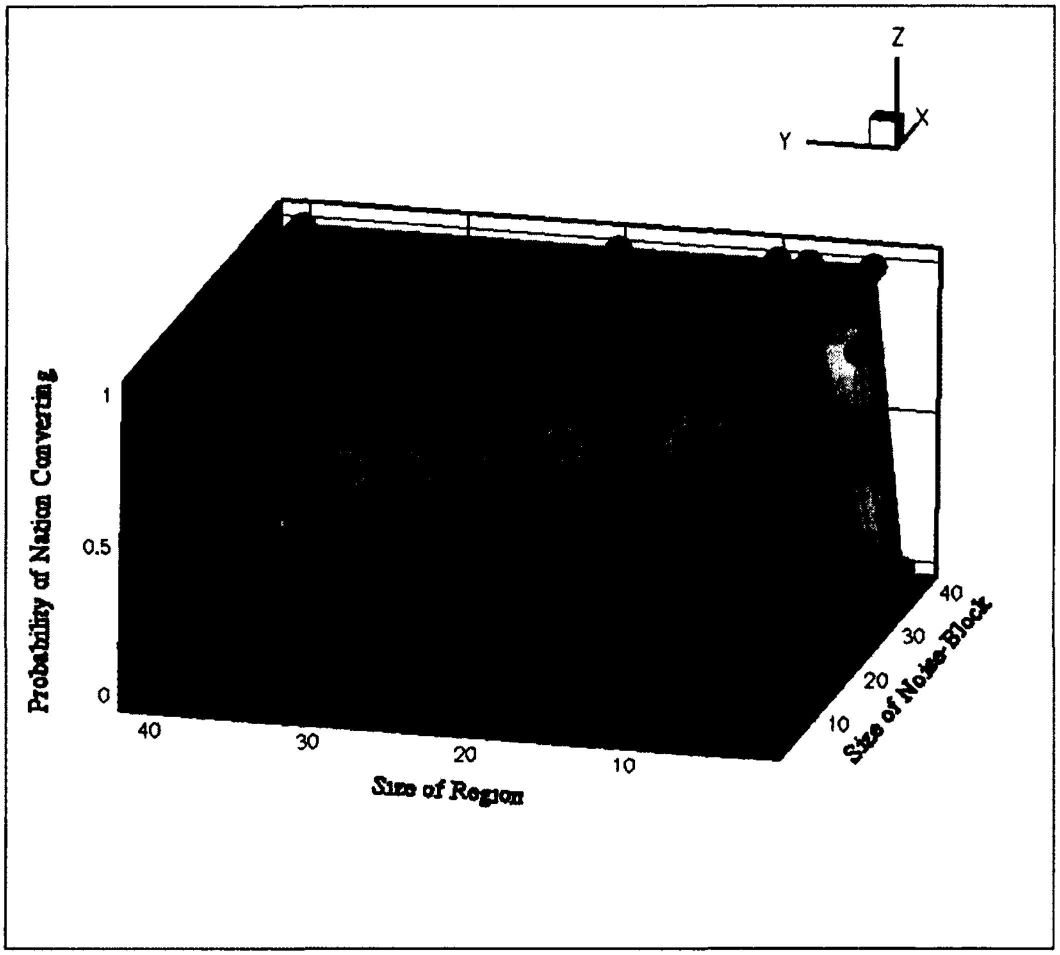


Figure 3.5: Combination of Three Different National Sizes(120×120,160×160 and 200×200)

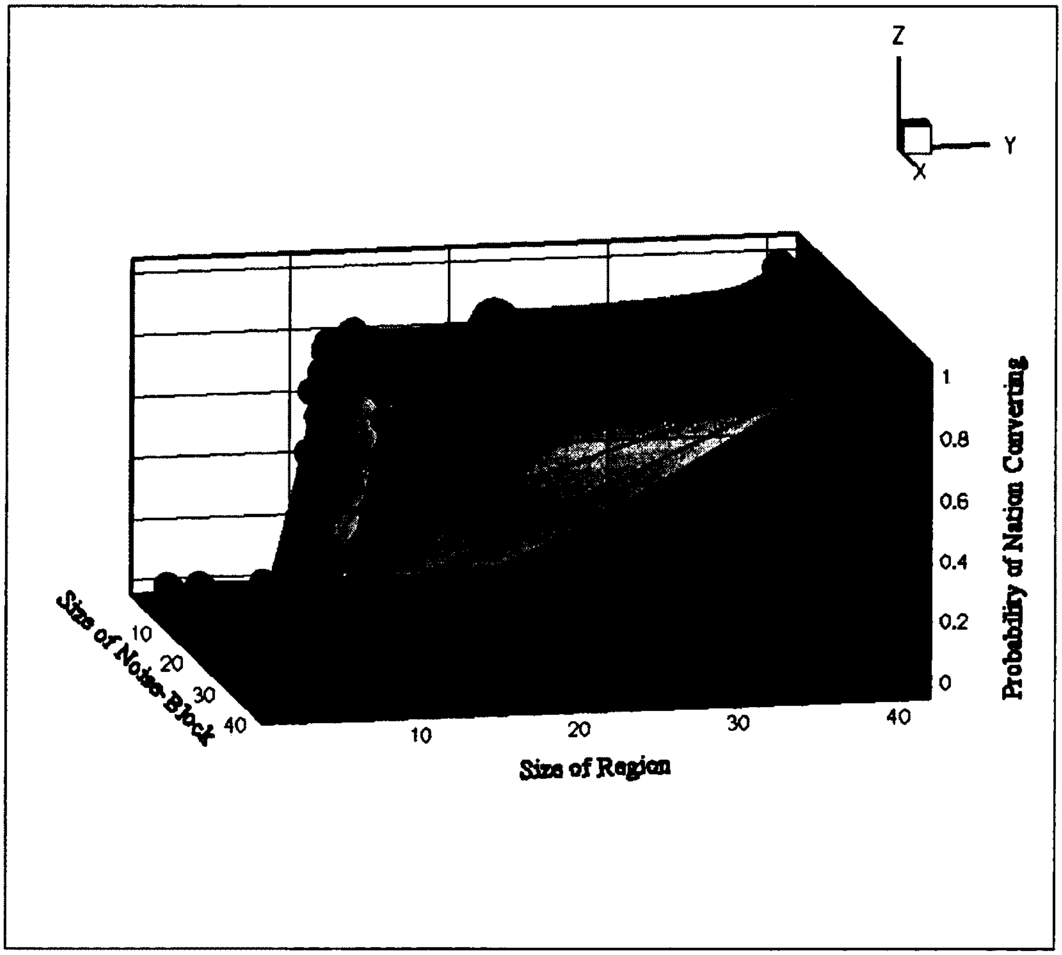


Figure 3.6: Combination of Three Different National Sizes(120 × 120,160 × 160 and 200 × 200) at a Different Angle

3.2 Sub-Regional Voting Model

3.2.1 Fixed National and Regional Size with Changing Sub-Regional Size and Noise-Block Size

In the sub-regional voting model, there were also two candidates (A and B). The proportion of votes for A and B were set the same as in the regional model. The same concentrated noise in regional model is applied in the sub-regional model. It is of interest to find out how different sub-regional sizes affect the performance of noise confinement in the sub-regional model. The same national sizes (120×120, 160×160 and 200×200) were selected for comparing the performance of noise confinement between the regional voting model and the sub-regional voting model. The greatest common divisor (40×40) between 120×120, 160×160 and 200×200 was selected as the regional size. It gave the most number of common sub-regional sizes (1×1, 2×2, 4×4, 8×8 and 10×10) among 120×120, 160×160 and 200×200 for comparing performance of noise confinement at different national sizes in the sub-regional model. The common divisor 20×20 was not selected here because the regional size needed to be at least 10 times that of the sub-regional size to use the binomial distribution to approximate the hypergeometric distribution as mentioned in Chapter 2. When the sub-regional size equalled 1×1, or the regional size in the sub-regional voting model, the performance of noise confinement was identical to the regional voting model. According to the following figures, the performance of noise confinement for the sub-regional voting model improved as the sub-regional size decreased until the size of the sub-region reached some certain threshold point. Then, it started decreasing to the performance of the regional voting model once the size of sub-region got below this threshold point. In the regional voting model, the national size of 200 × 200 gave a graph that indicated that there was 0 probability of the nation being converted at any of the tested sizes of concentrated noise-blocks for all the targeted regional sizes. If the sub-regional voting model was more stable than the regional voting model under noise influence, the same result would have been expected in the sub-regional voting model when the national size was set as 200 × 200.

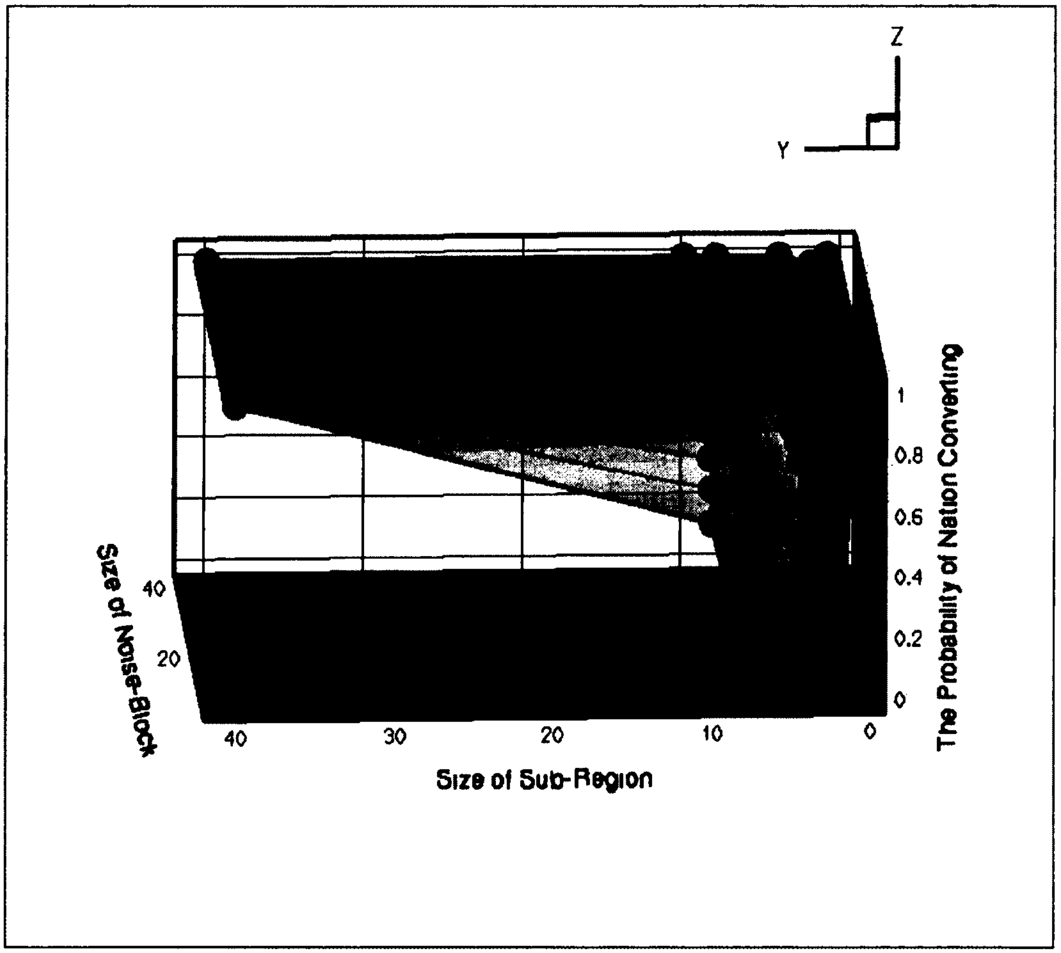


Figure 3.7: Sub-Regional Model with National Size of 120 × 120 and Regional Size of 40 × 40

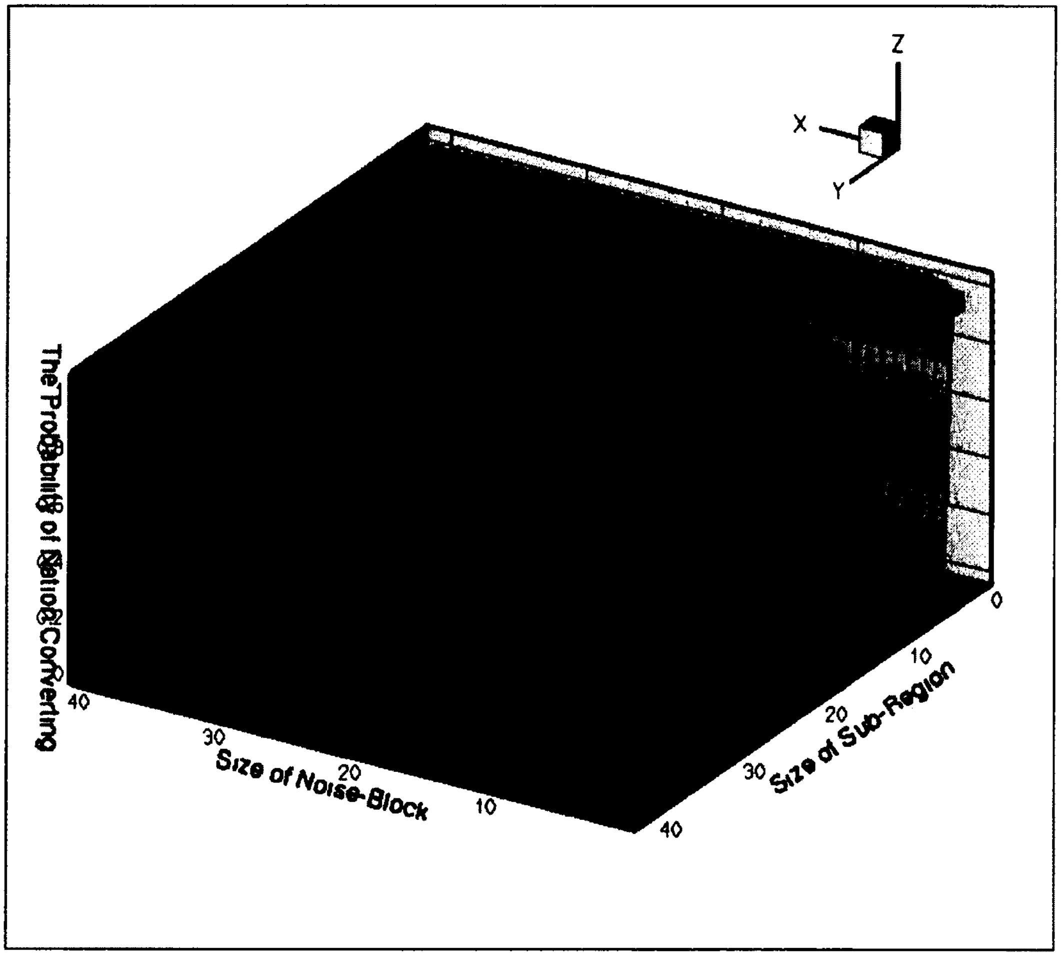


Figure 3.8: Sub-Regional Model with National Size of 120×120 and Regional Size of 40×40 at a Different Angle

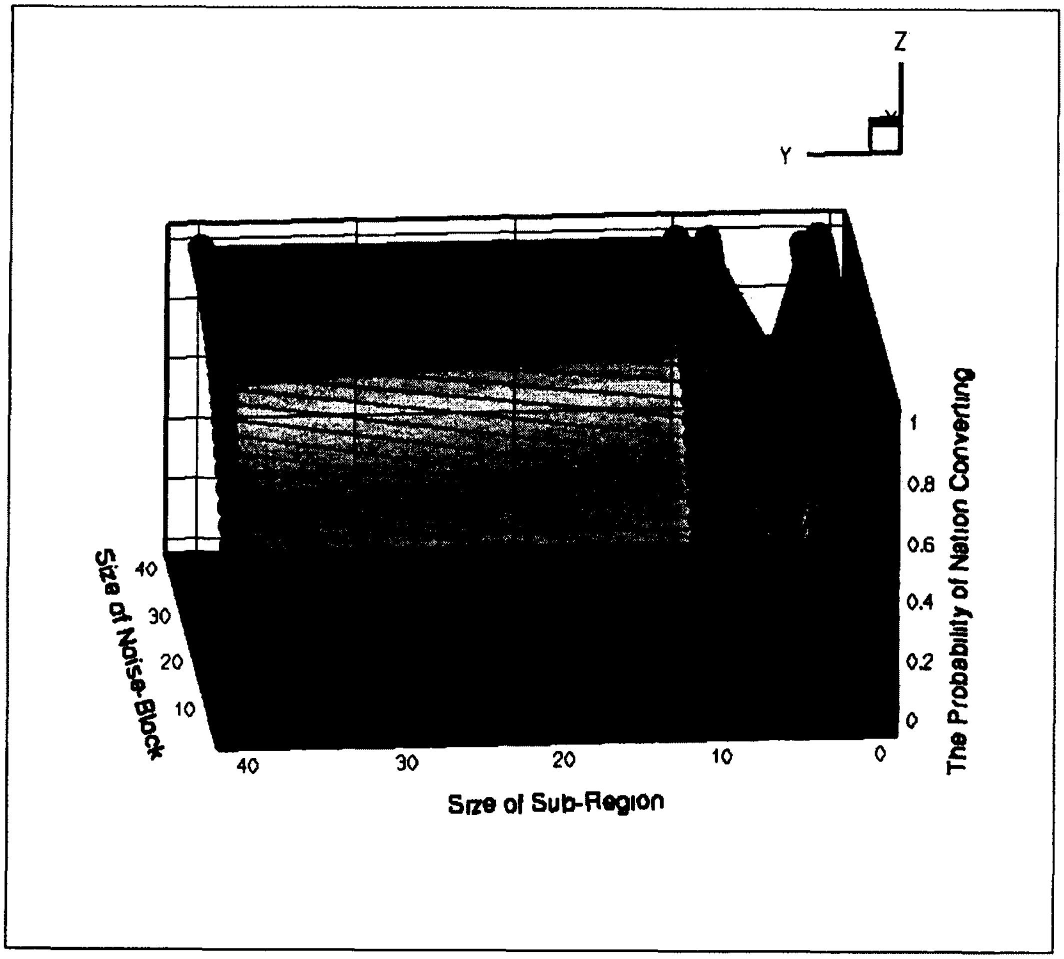


Figure 3.9: Sub-Regional Model with National Size of 160 × 160 and Regional Size of 40 × 40

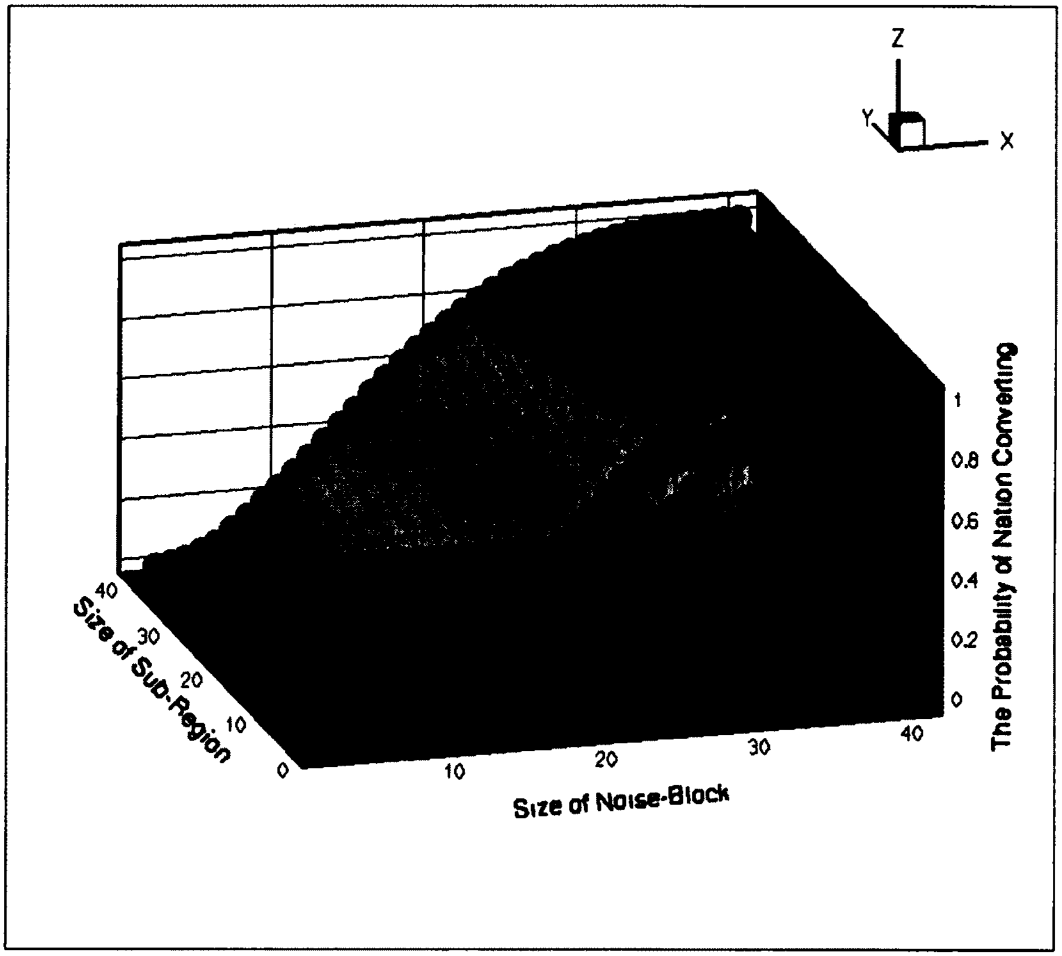


Figure 3.10: Sub-Regional Model with National Size of 160 × 160 and Regional Size of 40 × 40 at a Different Angle

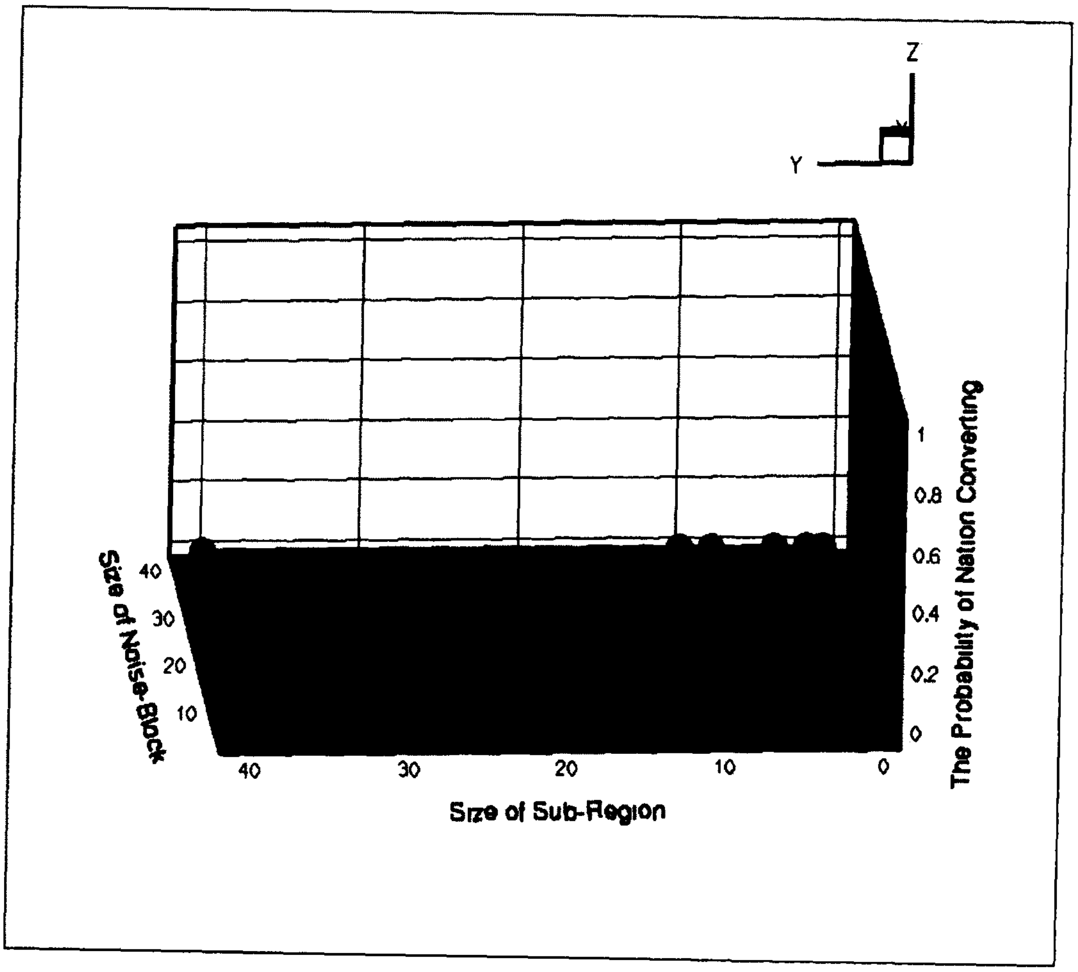


Figure 3.11: Sub-Regional Model with National Size of 200 × 200 and Regional Size of 40 × 40

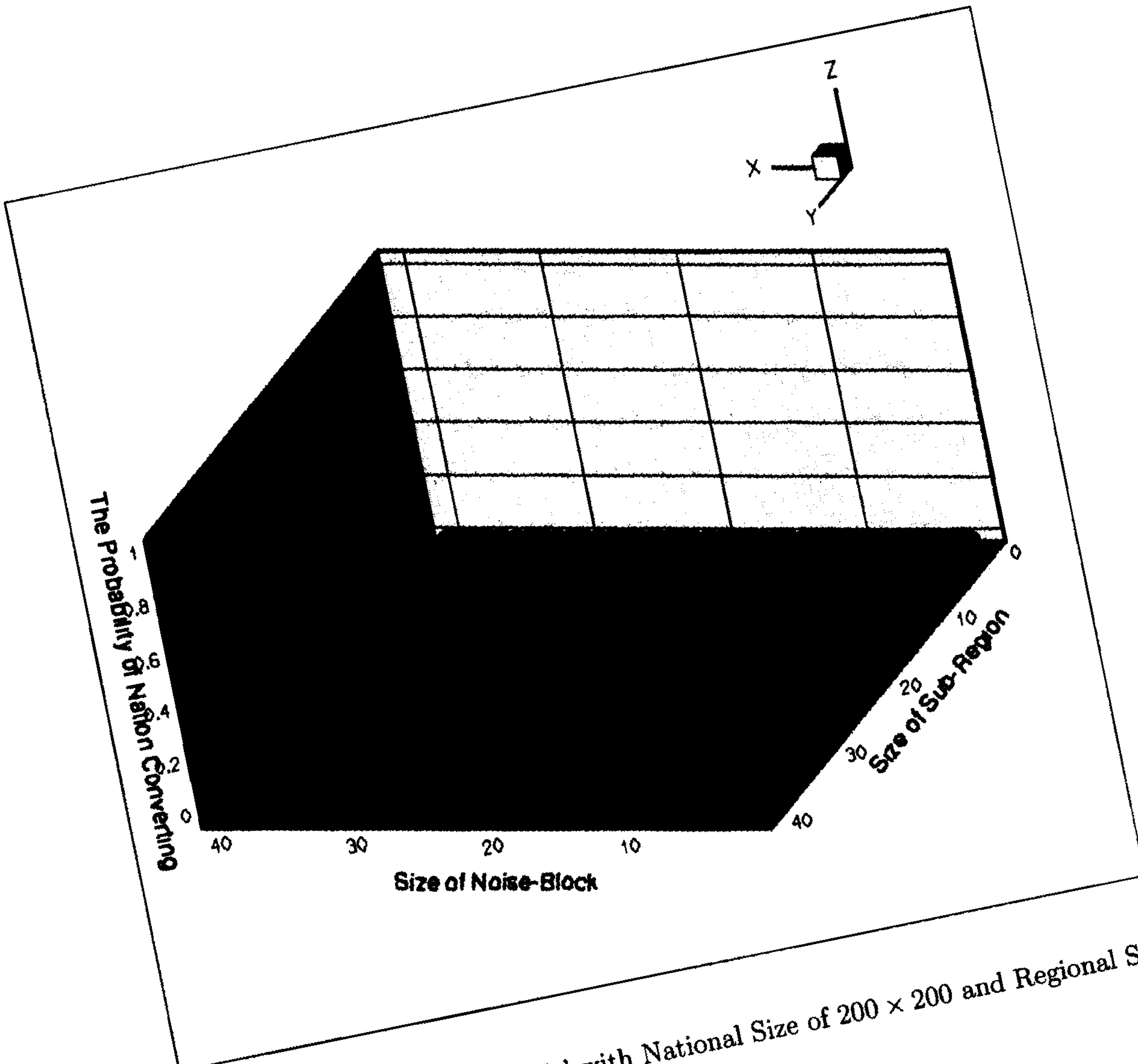


Figure 3.12: Sub-Regional Model with National Size of 200 × 200 and Regional Size of 40 × 40 at a Different Angle

3.2.2 Comparison between Models That Have Different National Sizes

Based on figures in section 3.2.1, the different national sizes demonstrated the different performances of noise confinement in the sub-regional voting model. Figures 3.13 and 3.14 indicate that the peformance of noise confinement improved as the national size increased in the sub-regional voting model. Moreover,the performance of noise confinement in the sub-regional voting model improved as the size of the noise-block decreased. Green stands for the national size of 120×120 in the sub-regional model, red stands for the national size of 160×160, and blue stands for the national size of 200×200.

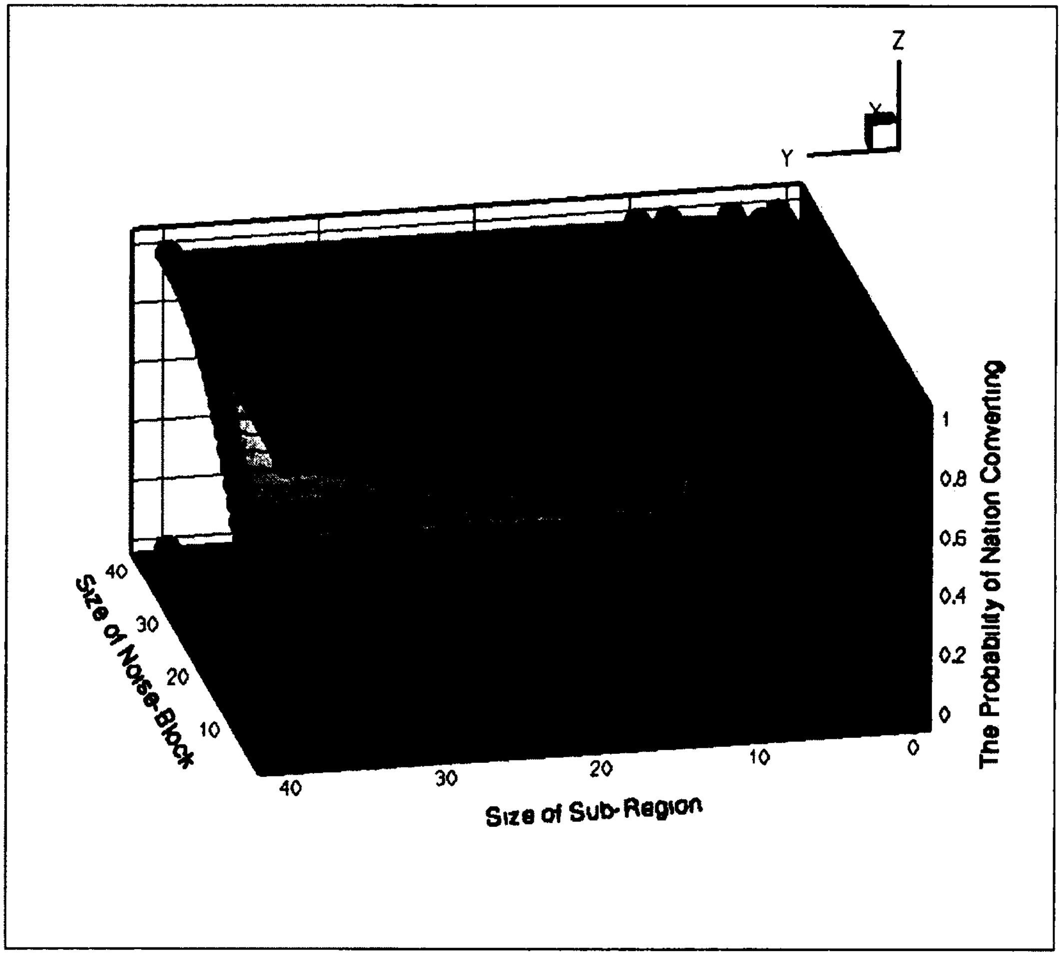


Figure 3.13: Combination of Three Different National Sizes(120×120,160×160 and 200×200)

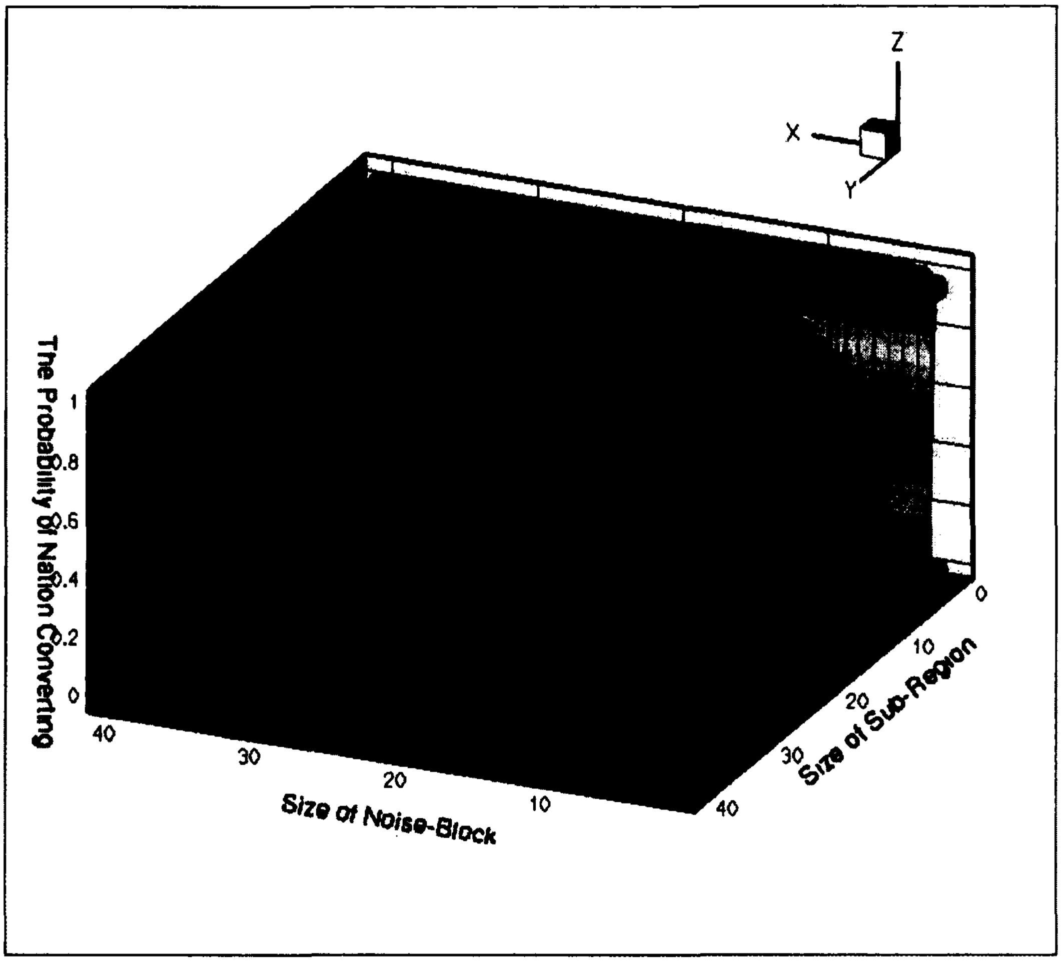


Figure 3.14: Combination of Three Different National Sizes (120 × 120,160 × 160 and 200 × 200) at a Different Angle

3.2.3 The Sub-regional Model with the Fixed National and Sub-regional Sizes but Difference in Regional Sizes

Three sub-regional models with national sizes "120 × 120","160 × 160" and "240 × 240" were tested respectively. Each model had a certain number of regions tested. The regional size selected for testing was a divisor of the national size and also a multiple of the sub-regional size. "2 × 2" and "4 × 4" were set as the fixed sub-regional sizes on both models of "120 × 120" and "160 × 160". "10 × 10" and

"20×20" were set as the sub-regional sizes on the model "240×240". In the model "120×120", there were nine regional sizes tested for the fixed sub-regional size of "2×2" and six regional sizes tested for the fixed sub-regional size of "4×4". In the model "160×160", there were seven regional sizes tested for the fixed sub-regional size of "2×2" and five regional sizes tested for the fixed sub-regional size of "4×4". In the model "240×240", both fixed sub-regional sizes "10×10" and "20×20" had four tested regional sizes.

The figures indicate that there exist certain regional sizes which gave the worst performance on noise confinement when the national size and sub-regional size were both fixed in the sub-regional model. On either side of this regional size, the performance of noise confinement got better as the size of region increased or decreased from this particular regional size. On the larger sizes of nations, this tendency became more obvious.

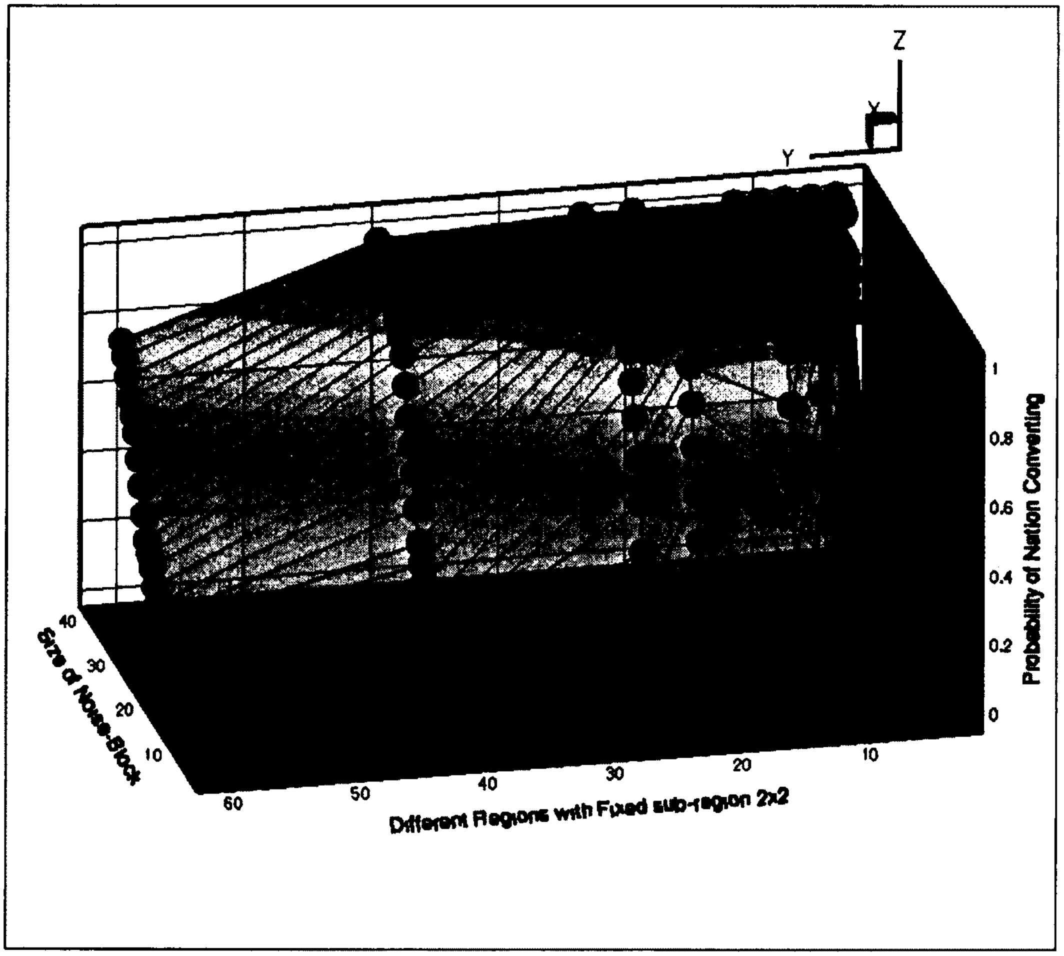


Figure 3.15: Fixed National Size-120 × 120 and Fixed Sub-Regional Size-2 × 2 with Various Regional Sizes (“4 × 4”,“6 × 6”,“8 × 8”,“10 × 10”,“12 × 12”,“20 × 20”,“24 × 24”,“40 × 40” and “60 × 60”)

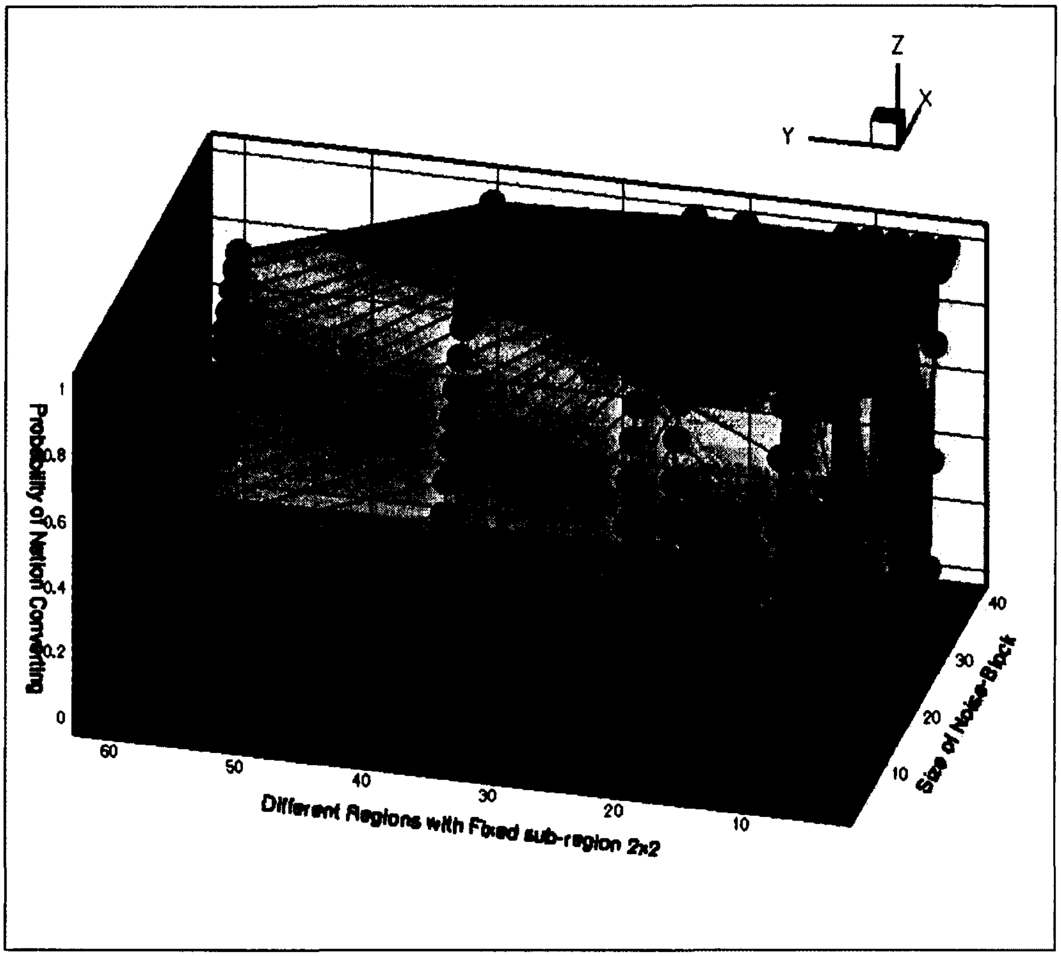


Figure 3.16: Fixed National Size-120 × 120 and Fixed Sub-Regional Size-2 × 2 with Various Regional Sizes ("4 × 4","6 × 6","8 × 8","10 × 10","12 × 12","20 × 20","24 × 24","40 × 40" and "60 × 60") at a Different Angle

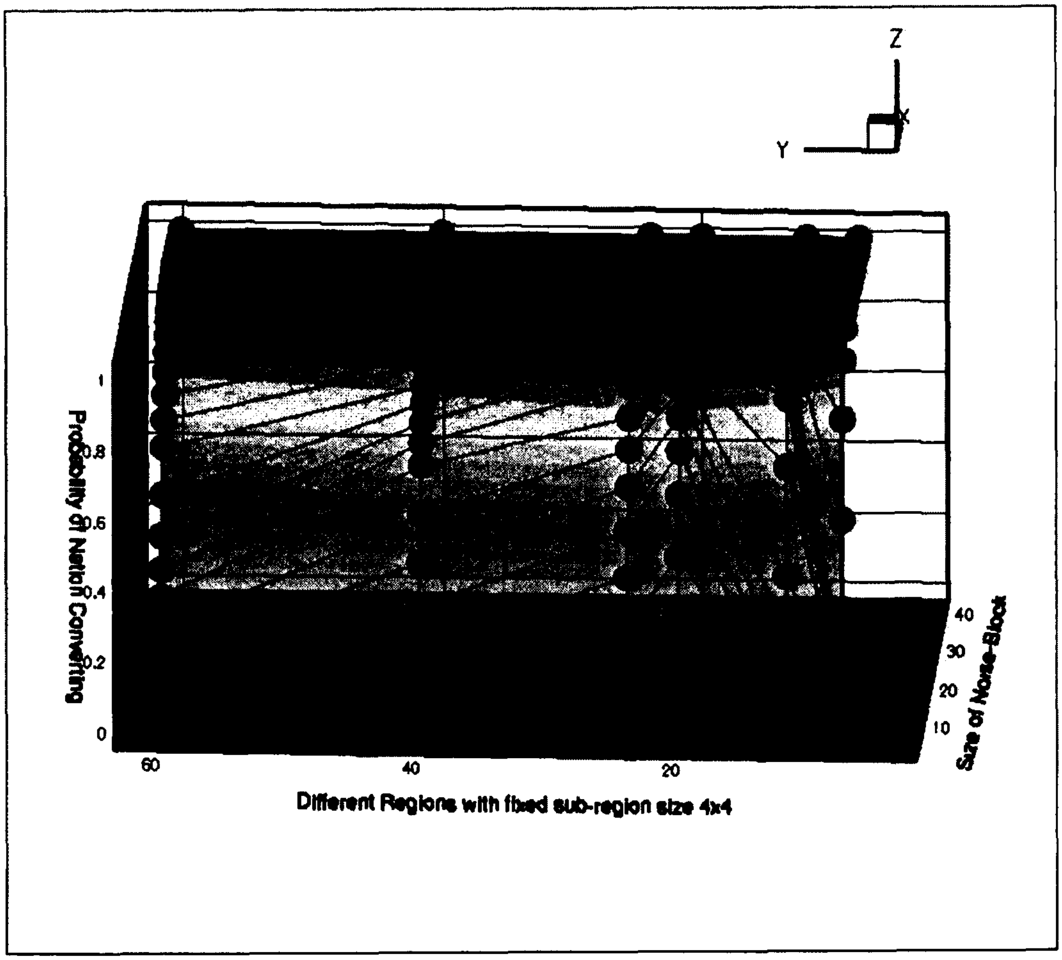


Figure 3.17: Fixed National Size 120 × 120 and Fixed Sub-Regional Size 4 × 4 with Various Regional Sizes ("8 × 8","12 × 12","20 × 20","24 × 24","40 × 40", and "60 × 60")

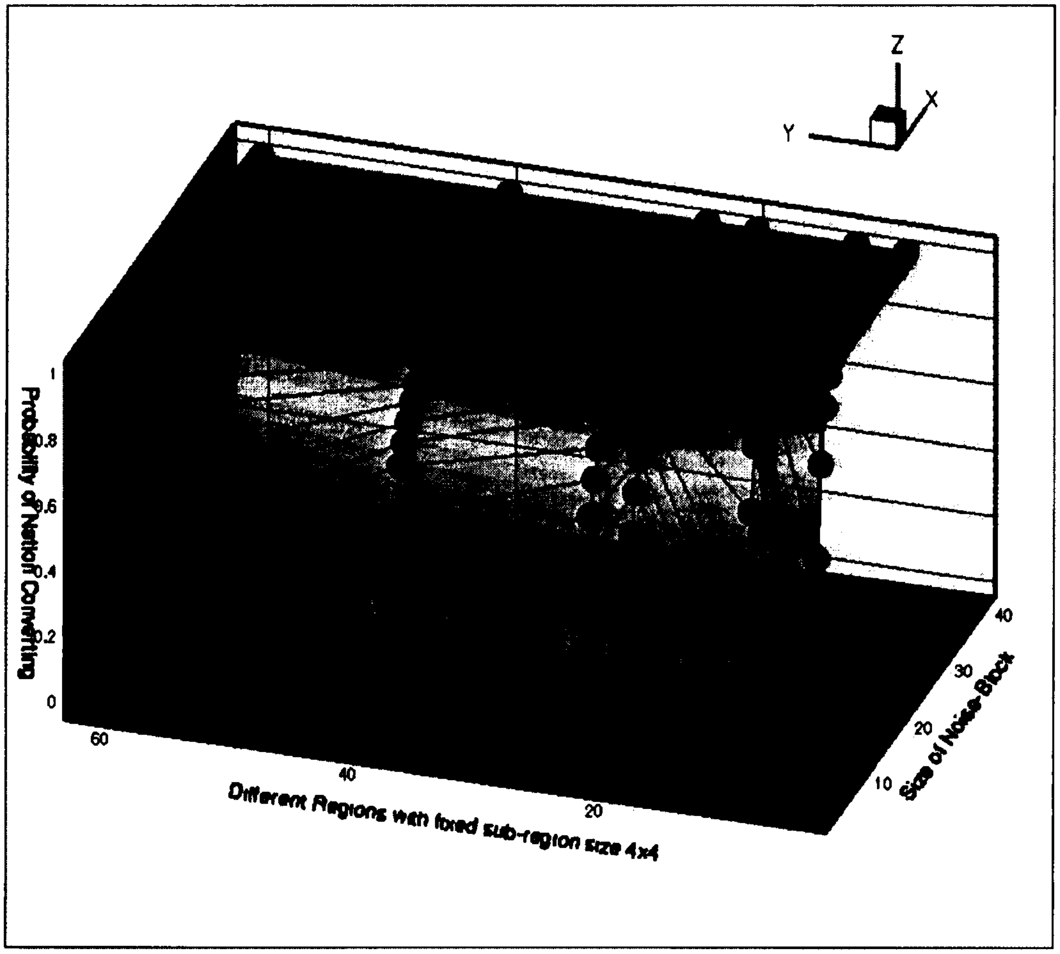


Figure 3.18: Fixed National Size 120 × 120 and Fixed Sub-Regional Size 4 × 4 with Various Regional Sizes ("8 × 8", "12 × 12", "20 × 20", "24 × 24", "40 × 40", and "60 × 60") at a Different Angle

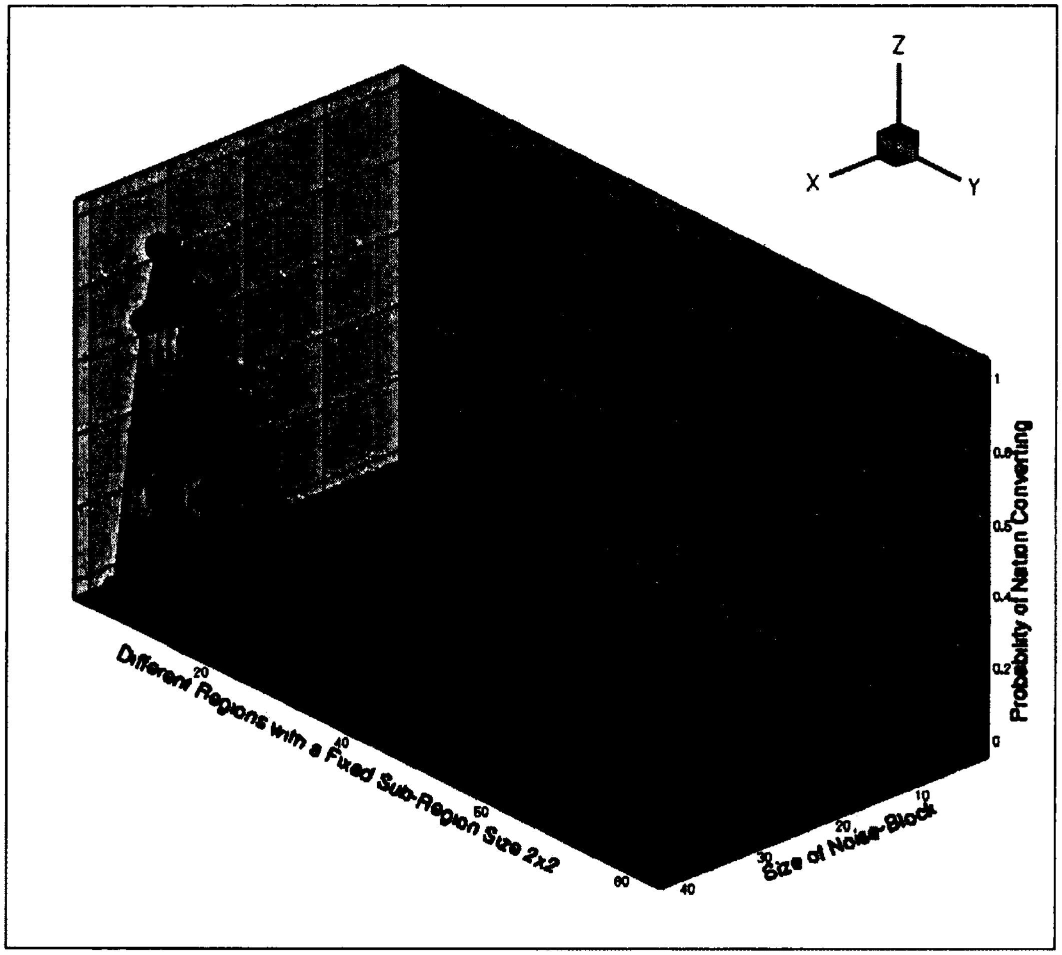


Figure 3.19: Fixed National Size 160 × 160 and Fixed Sub-Regional Size 2 × 2 with Various Regional Sizes ("4 × 4","8 × 8","10 × 10","16 × 16","20 × 20","40 × 40", and "80 × 80")

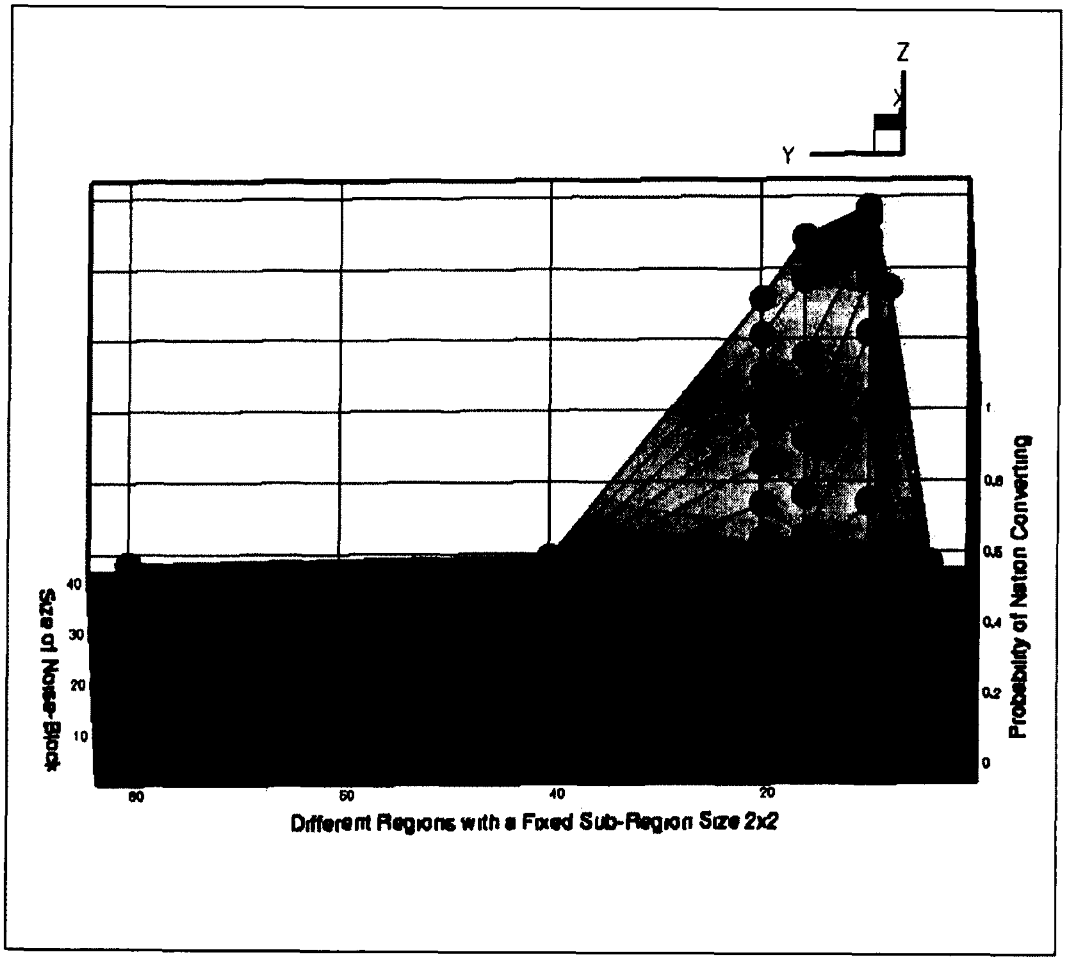


Figure 3.20: Fixed National Size 160 × 160 and Fixed Sub-Regional Size 2 × 2 with Various Regional Sizes at (“4 × 4”,“8 × 8”,“10 × 10”,“16 × 16”,“20 × 20”,“40 × 40”, and “80 × 80”) a Different Angle

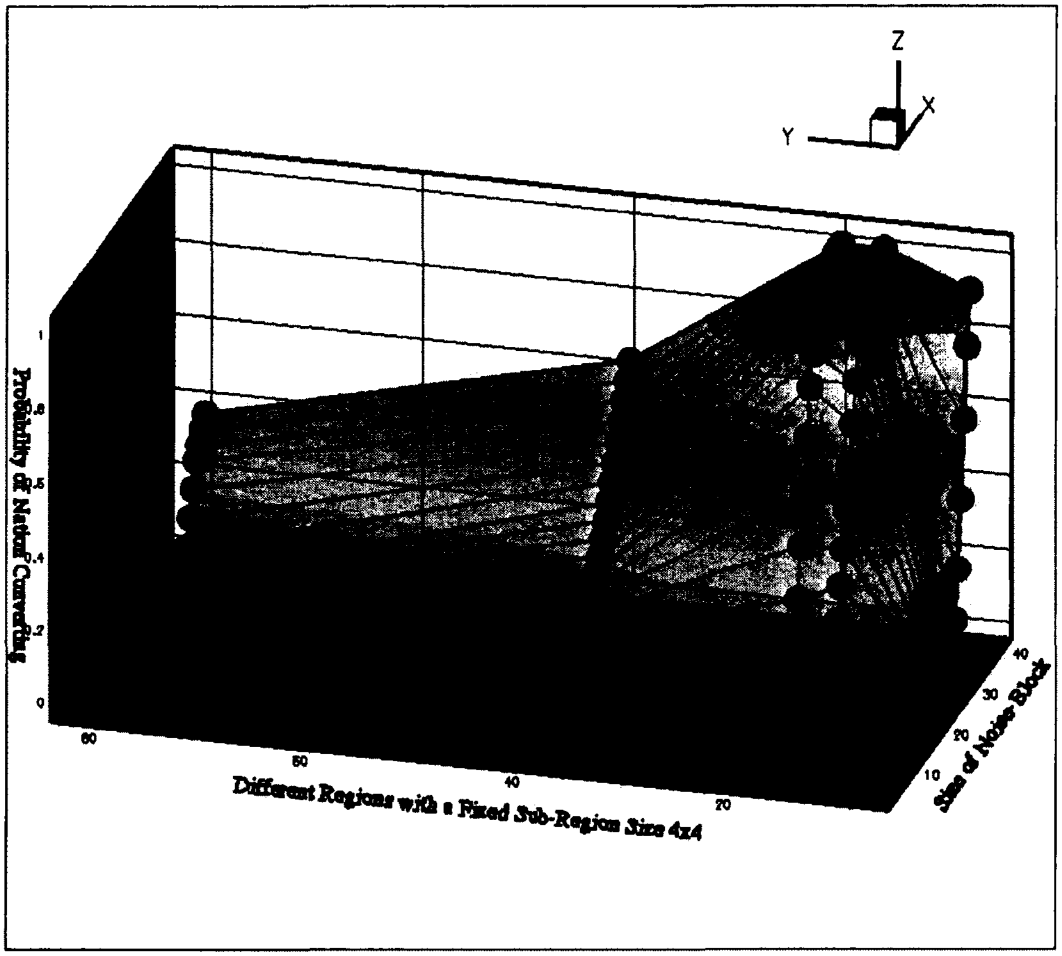


Figure 3.21: Fixed National Size 160 × 160 and Fixed Sub-Regional Size 4 × 4 with Various Regional Sizes (“8 × 8”, “16 × 16”, “20 × 20”, “40 × 40”, and “80 × 80”)

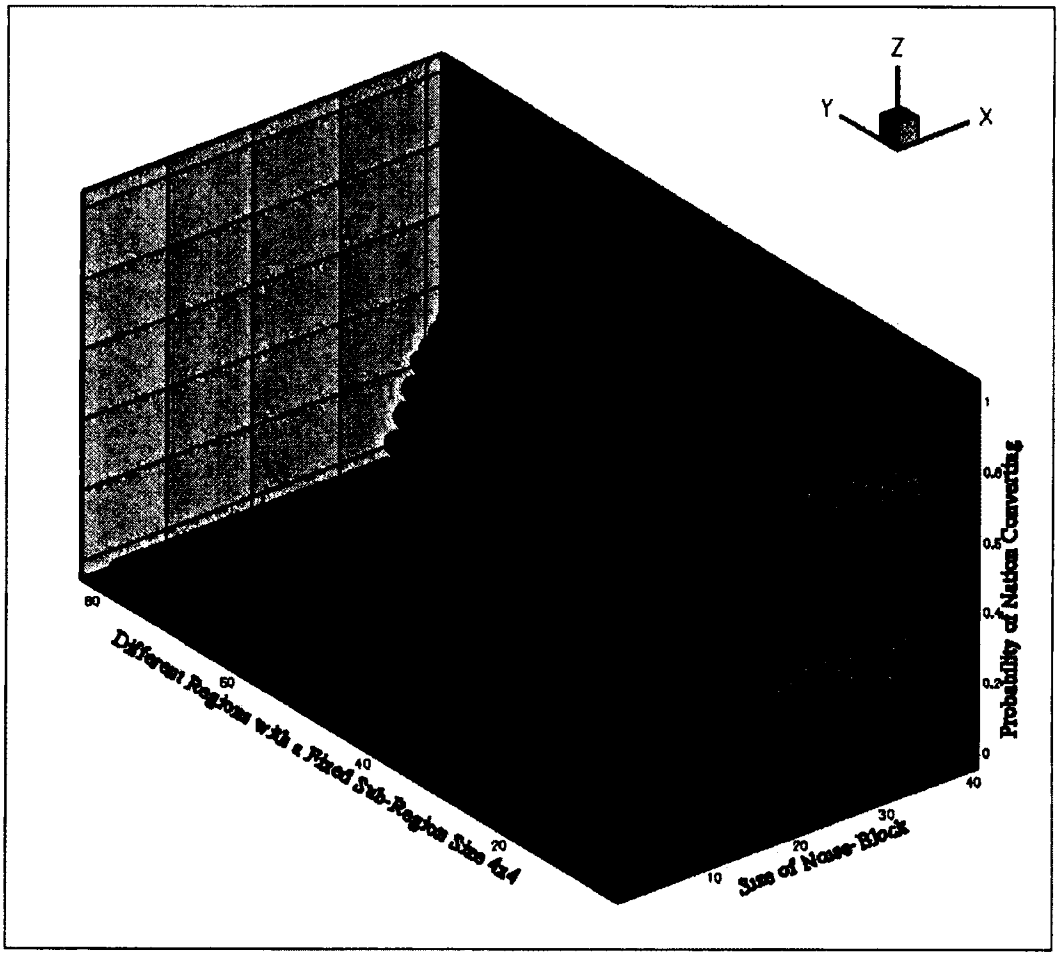


Figure 3.22: Fixed National Size 160 × 160 and Fixed Sub-Regional Size 4 × 4 with Various Regional Sizes ("8 × 8", "16 × 16", "20 × 20", "40 × 40", and "80 × 80") at a Different Angle

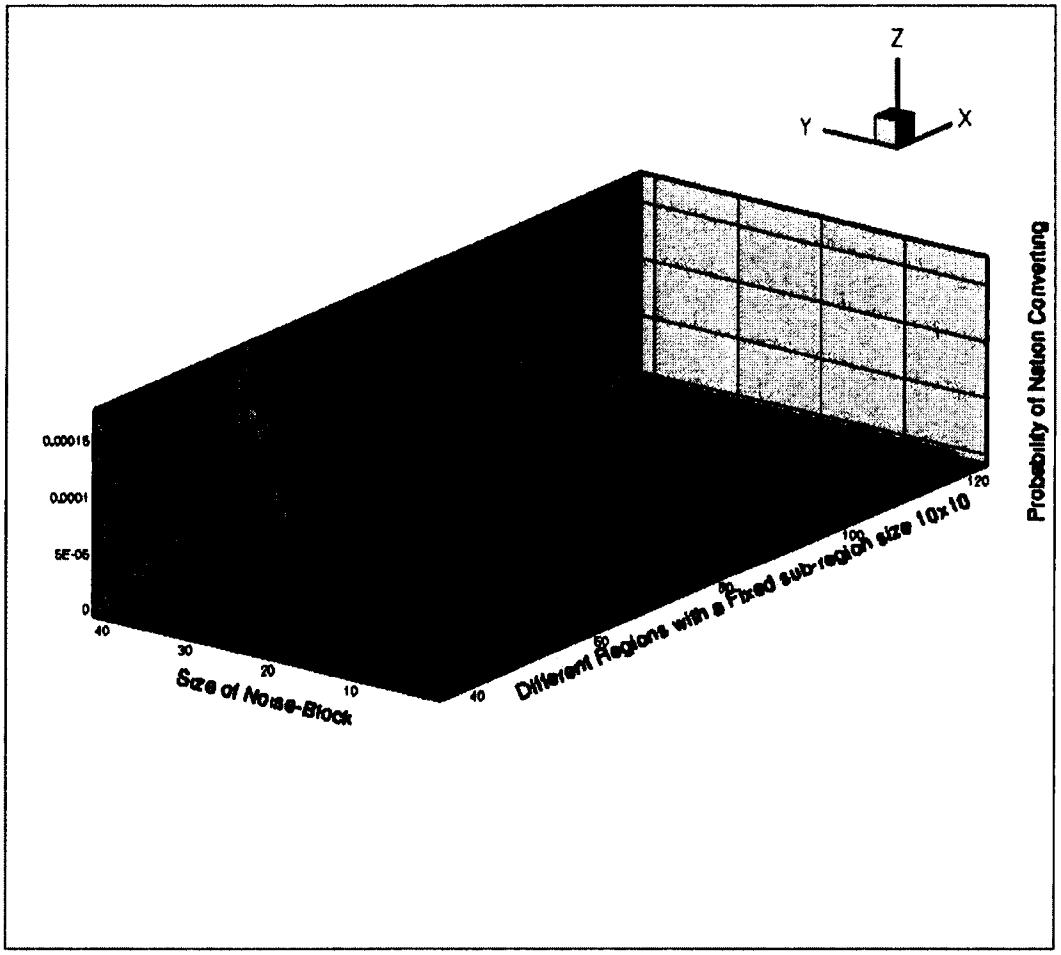


Figure 3.23: Fixed National Size 240×240 and Fixed Sub-Regional Size 10×10 with Various Regional Sizes ("40×40", "60×60", "80×80", and "120×120")

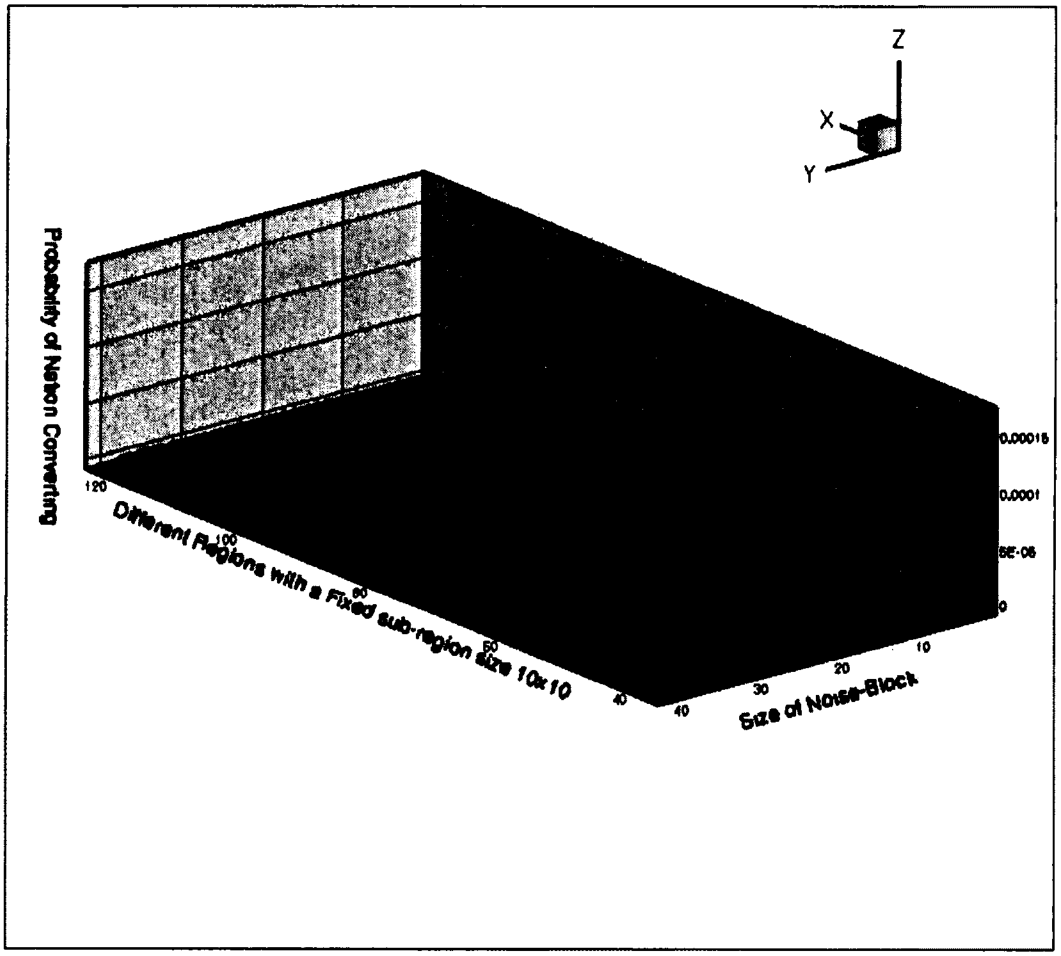


Figure 3.24: Fixed National Size 240×240 and Fixed Sub-Regional Size 10×10 with Various Regional Sizes ("40×40", "60×60", "80×80", and "120×120") at a different Angle

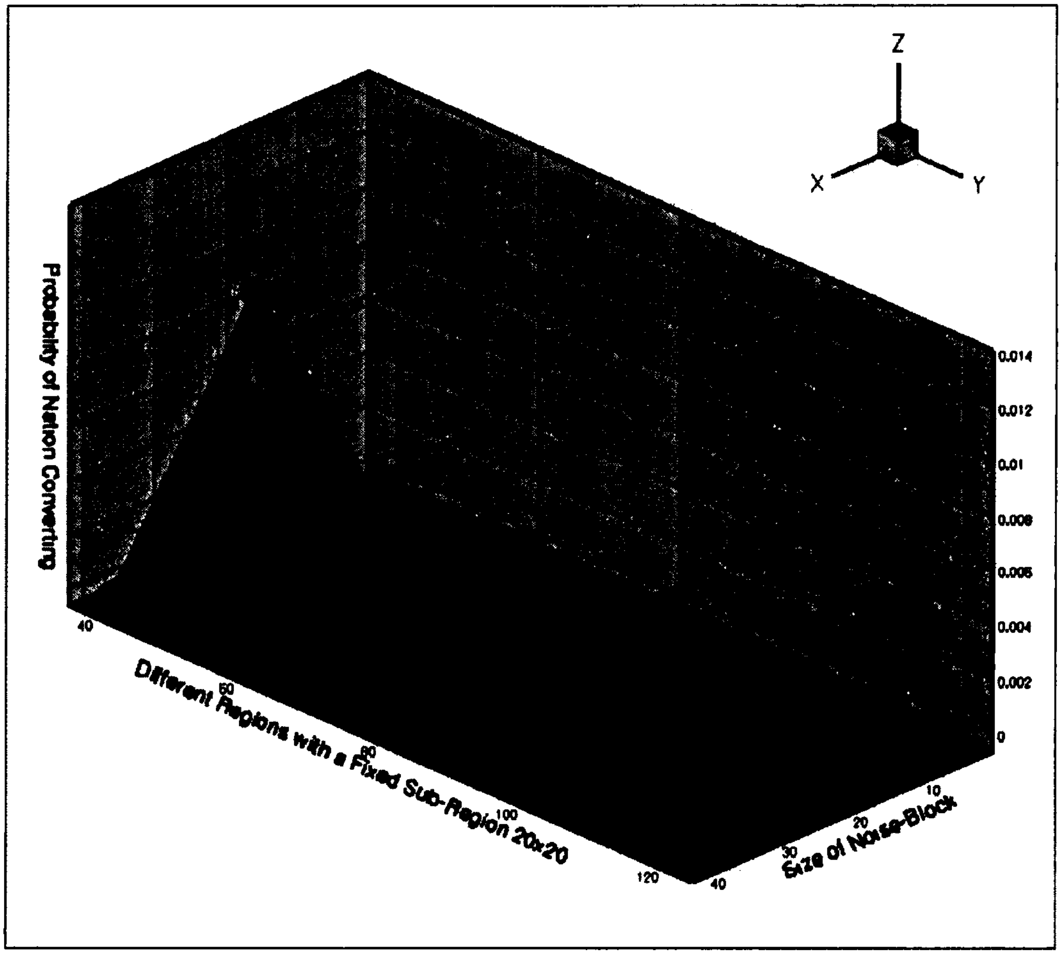


Figure 3.25: Fixed National Size 240 × 240 and Fixed Sub-Regional Size 20 × 20 with Various Regional Sizes ("40 × 40", "60 × 60", "80 × 80", and "120 × 120")

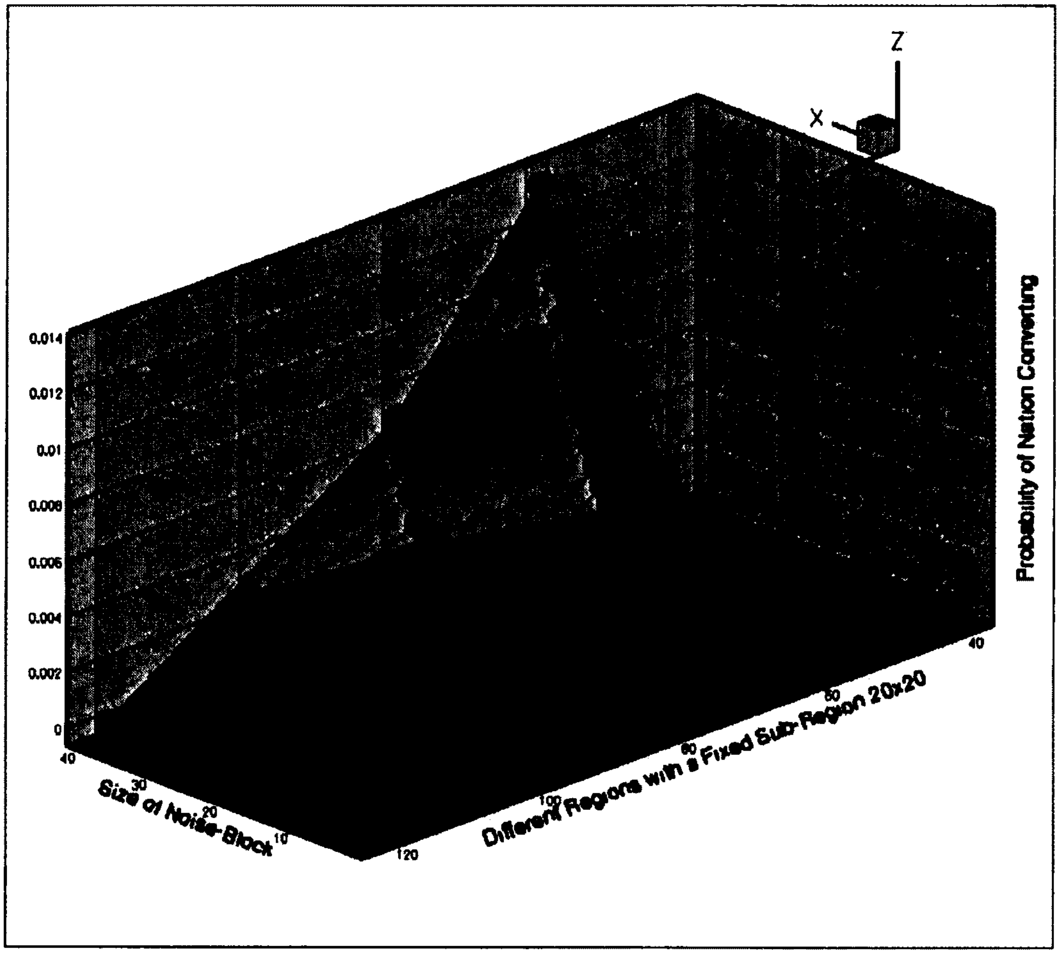


Figure 3.26: Fixed National Size 240 × 240 and Fixed Sub-Regional Size 20 × 20 with Various Regional Sizes ("40 × 40", "60 × 60", "80 × 80", and "120 × 120") at a Diffferent Angle

Chapter 4

Conclusion and Extended Work

4.1 Conclusion

There are four conclusions drawn from this study.

First, both the regional and the sub-regional voting schemes tend to be much less stable when the size of noise blocks increased. The sub-regional voting scheme is more stable than the regional voting scheme under certain size of noise blocks. However, if the sizes of the noise-blocks goes beyond this threshold, then the national election result will be converted from A to B under both voting schemes.

Second, both schemes become more stable as the size of nations increases. If the size of noise blocks are significantly smaller than the size of nations, then the nation is unlikely to be converted.

Third, the performance of noise confinement in the sub-regional voting scheme depends on the size of the sub-regions. In the sub-regional scheme, if the size of nations and the size of regions are fixed, then a certain size of sub-regions exists which contains the most concentrated noise.

Fourth, if the size of nations and the size of sub-regions are fixed, the performance on the noise confinement relies on the size of regions being chosen in the sub-regional voting scheme. In this case, a certain size of regions exists which confines the concentrated noise at the lowest amount.

4.2 Extended Work

1. In the Model, the only noise considered is concentrated noise. Another type of noise (white noise) could be taken into account. White noise is a type of noise that affects the whole nation uniformly and randomly. This noise can be treated as a factor that directly changes the probability of one voter voting for A or for B.

2. In the process of facial recognition feature matching, each facial image can be broken into regions and sub-regions. This could be done in the same way that this model presents. This will allow for a better facial recognition rate.

3. It would be interesting to find out whether or not the noise confinement could be improved by recursively partitioning each sub-region into even smaller sub-regions in the sub-regional voting scheme.

Appendix A

The Binomial Approximation to The Hypergeometric Distribution

When the national size is very big, the hypergeometric distribution would take a very long time to compute. Therefore, the binomial distribution is used to approximate the hyergeometric distribution in order to optimize the calculation[7]. The binomial distribution gives the probability of x successes in n trials where the probability of success is p. The hypergeometric distribution is very much the same as the binomial distribution, except the hypergeometric distribution's probability changes with each trial. In other words, the only difference between hypergeometric distribution and binomial distribution is that the bionomial distribution samples with replacement and the hypergeometric distribution samples without replacement. The probability mass function for the binomial distribution is

$$P(X = k) = \binom{n}{k} P^k (1 - P)^{n-k}. \tag{A.1}$$

It gives the probability of getting exactly k successes in n trials. When a population size is large enough, the hypergeometric probability mass function is almost the same as the binomial probability mass function. In practice, this means that the binomial distribution can be used to approximate the hypergeometric distribution. Some tests were done to show that if the national size is at least 10 times of the sample size ($N \geq 10n$), then the hypergeometric distribution can be replaced with binomial distribution while remaining accurate. The test done to verify this approximation is:

$$P(6 \leqslant k \leqslant 10) = \sum_{k=6}^{10} \frac{\binom{60}{k}\binom{40}{10-k}}{\binom{100}{10}} = 0.6385503711$$

$$P(6 \leqslant k \leqslant 10) = \sum_{k=6}^{10} \binom{10}{k} 0.6^k (0.4)^{10-k} = 0.6331032576.$$

In this test, the chosen sample size was 10, which is $\frac{1}{10}$ of size of the nation. If there are 60 voters voting for A, out of a nation size of 100, then the rest of the 40 remaining voters would be voting for B. The chance of more than half of the voters voting for A in a region of size of 10 can be calculated by the hypergeometric probability mass function and by the binomial probability mass function respectively, demonstrated above. It gives a fairly accurate approximation. The results remain the same until the third decimal.

The probability of a sub-region voting for A and the probability of a sub-region voting B:

$$\alpha_s = \sum_{k=\lceil(\frac{s_l \times s_w}{2})\rceil}^{sl \times sw} \binom{s_l \times s_w}{k} (\alpha)^k (\beta)^{s_l \times s_w - k}; \tag{A.2}$$

$$\beta_s = 1 - \alpha_s. \tag{A.3}$$

The probability of a region voting for A and the probability of a region voting for B:

$$\alpha_r = \sum_{k=\lceil(\frac{r_l \times r_w}{2})\rceil}^{r_l \times r_w} \binom{r_l \times r_w}{k} (\alpha)^k (\beta)^{r_l \times r_w - k}; \tag{A.4}$$

$$\beta_r = 1 - \alpha_r. \tag{A.5}$$

Appendix B

The Normal Approximation to The Binomial Distribution

If a sub-region or region size is very large, the calculation would take a very long time by using the binomial mass probability function. Therefore, a second stage of approximation is applied, which is using the normal distribution to approximate the binomial distribution[2]. It is generally known that the normal distribution is a close approximation to the binomial distribution when $n \times p \geq 5$ and $n \times q \geq 5$. $\varphi = n \times p$ stands for the mean of the binomial distribution, and $\sigma = \sqrt{n \times p \times q}$ stands for the standard deviation of the binomial distribution. n represents the sample size, and p represents the probability of success in the event that is of interest. By applying the probability density function of the normal distribution, the probability of a sub-region voting for A (α_s) or a region voting for A (α_r) and the probability of a sub-region voting for B(β_s) or a region voting for

$B(\beta_r)$ can be calculated. The probability density function for normal distribution is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}. \tag{B.1}$$

The expectation (mean) of the distribution and the standard deviation of the distribution are:

$$\mu = s_w \times s_l \times (\alpha); \tag{B.2}$$

$$\sigma = \sqrt{s_w \times s_l \times (\alpha) \times (\beta)}. \tag{B.3}$$

In the normal distribution, probability is calculated by taking the integral of the probability density function. The probability of sub-region voting for A and the probability of sub-region voting for B:

$$\alpha_s = \int_{\lceil \frac{sw \times sl}{2} \rceil}^{sw \times sl} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx; \tag{B.4}$$

$$\beta_s = 1 - \alpha_s. \tag{B.5}$$

The probability of a region voting for A and the probability of a region voting for B:

$$\alpha_r = \int_{\lceil \frac{rw \times rl}{2} \rceil}^{r_w \times r_l} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx; \tag{B.6}$$

$$\beta_r = 1 - \alpha_r. \tag{B.7}$$

The expectation (mean) of the distribution and the standard deviation of the distribution are:

$$\mu = r_w \times r_l \times (\alpha); \tag{B.8}$$

$$\sigma = \sqrt{r_w \times r_l \times (\alpha) \times (\beta)}. \tag{B.9}$$

A test can be done to verify this approximation. For a sample size $n = 10$ with probability of success $p = 0.5$, the binomial distribution plot is displayed in figure 2.2, and the normal distribution plot on top of this binomial plot is displayed figure 2.3. A really good approximation is achieved. There is only one point that is slightly off the normal distribution curve in figure 2.3.

```
n := 10 : p := 0.5 :
with(Statistics) :
X := RandomVariable(Binomial(n, p)) :

A := seq([n, ProbabilityFunction(X, n)], n = 0 ..10) :
plot([A], style = point);
```

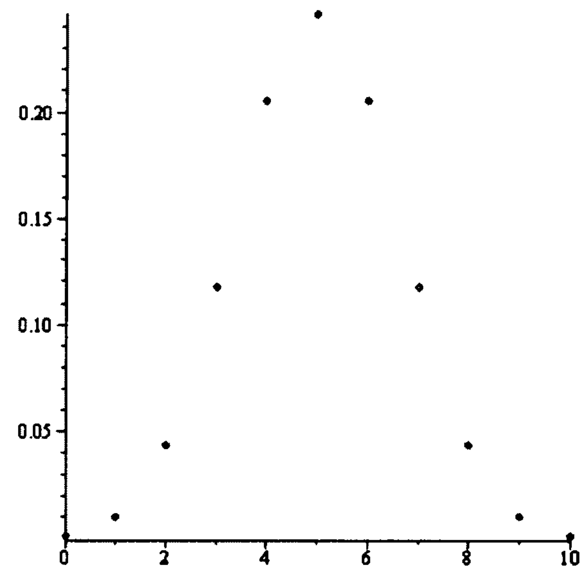


Figure B.1: The Binomial Distribution Plot

```
AA := plot([A], style = point) :
μ := n·p:
σ := sqrt(n·p·(1 − p)) :
Y := RandomVariable(Normal(μ, σ)) :
BB := plot(PDF(Y, x), x = 0..10) :

with(plots) :
display({AA, BB});
```

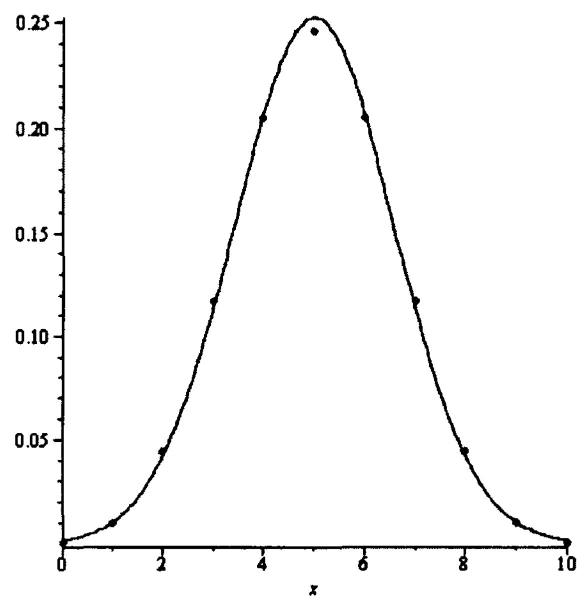


Figure B.2: Demonstration of The Normal Distribution approximating The Binomial Distribution

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